# On stochastic-user-equilibrium-based day-to-day dynamics

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## Abstract

Researchers have proposed many different concepts and models to study the day-to-day dynamics. Some models explicitly model travelers' perceiving and learning on travel costs, while some other models do not explicitly consider the travel cost perception but formulate the dynamics of flows as the functions of flows and measured travel costs (which are determined by flows). This paper investigates the interconnection between these two types of day-to-day models, in particular those models whose fixed points are stochastic user equilibrium. Specifically, a widely used day-to-day model which combines exponential-smoothing learning and logit stochastic network loading (called the *logit-ESL* model in this paper) is proved to be equivalent to a model based purely on flows, which is the logit-based extension of the first-in-first-out dynamic in Jin (2007). Via this equivalent form, the logit-ESL model is proved to be globally stable under nonseparable and monotone travel cost functions. Meanwhile, the model in Cantarella and Cascetta (1995) is shown to be equivalent to a second-order dynamic incorporating purely flows and is proved to be globally stable under separable link cost functions. Further, other discrete choice models such as C-logit, path-size logit and weibit are introduced into the logit-ESL model, leading to several new day-to-day models, which are also proved to be globally stable under different conditions.

*Keywords: day-to-day dynamics; stochastic user equilibrium; logit; first-in-first-out (FIFO) dynamic; weibit*

## 1. Introduction

As a complement to the user equilibrium (UE) theory to model and understand the transportation system, the day-to-day dynamics look into the evolution of traffic flows under travelers' repeated route choice behavior from day to day.

The day-to-day dynamics are modelled as dynamical systems. The independent variable of the day-to-day dynamical systems is the calendar time, e.g., a day or an epoch such as the morning peak of working days. The independent variable (calendar time) of a day-to-day model is discrete in nature; however, both the discrete-time and continuous-time models are widely used in the literature. Simply put, the continuous-time model is an approximation of the discrete-time model, while the stability of the former is easier to analyze; meanwhile, the stability of a continuous-time system can partially imply the stability of its discrete-time counterpart: if a continuous-time system is stable, then there exists a sufficiently small discrete step size such that the corresponding discrete-time model is stable. Deeper and more detailed discussions on the difference and relationship of the discrete-time and continuous-time day-to-day models can be found in Watling (1999) and Cantarella and

Watling (2016). For the convenience of theoretical analysis, this paper focuses on continuous-time models.

In a day-to-day dynamic, the state variables represent the status of the transportation system and evolve according to certain rules. The normally used state variables are flows and perceived costs, either on links or on paths. These state variables have been used since the earliest research on day-to-day dynamics, viz. Horowitz (1984) and Smith (1984).

Horowitz (1984) assumed that the evolution of network flows is driven by travelers' learning behavior regarding their perception/prediction on future travel costs. To illustrate, we assume a network (Figure 1) with one origin-destination (OD) pair and two parallel links/paths numbered 1 and 2. Travelers are assumed to have their perception $p_k$ on the travel cost of each path $k \in \{1,2\}$, which may be different from the actual travel costs. The flow $f_k$ on path $k$ is determined by the perceived costs on both paths, i.e., $f_k = f_k\left(p_1, p_2\right)$, through some network loading rule such as the logit rule. The actual travel cost $c_k$ on each path $k$ is decided by the cost-flow relationship, i.e., $c_k = c_k\left(f_1, f_2\right)$. After knowing the actual costs, travelers will update their perception on each path, based on some learning rule such as the exponential smoothing:

$$\dot{p}_k = \alpha\left(c_k - p_k\right), \quad k = 1,2$$

where $\dot{p}_k \triangleq \mathrm{d}p_k / \mathrm{d}t$ is the derivative of $p_k$ with respect to the calendar time $t$, and $\alpha > 0$ is a constant parameter.

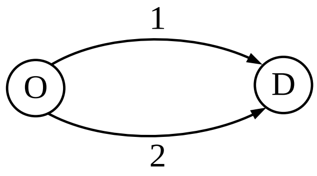


**Figure 1.** Network with one OD pair and two parallel links

Following Horowitz (1984)'s idea, many day-to-day models have used perceived costs as the state variables to consider travelers' perceiving/predicting and learning behavior on travel costs. A typical structure of such models is illustrated in Figure 2: the flow on each path is determined by travelers' perceived costs on all paths through certain network loading rules; once travelers experience/know the new path costs, they will update their perception through a learning process by combining the old perceived costs with the new actual costs. In the literature, most of these models have stochastic UE (SUE) as their fixed points due to the stochastic network loading (SNL) procedure adopted for deciding the path flows (e.g., Bie and Lo, 2010; Bifulco et al., 2016; Cantarella, 2013; Cantarella and Cascetta, 1995; Watling, 1999; Xiao and Lo, 2015; Ye and Yang, 2013; Ye et al., 2021), with a few exceptions which consider the dynamics of both flows and perceived costs and converge to Wardrop's UE (He and Liu, 2012; Xiao et al., 2016).

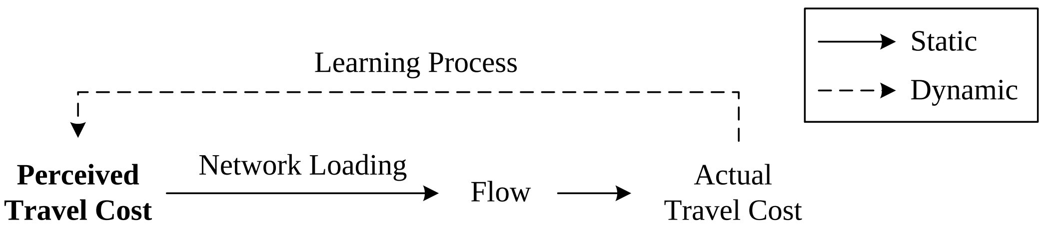


**Figure 2.** Flow chart of a typical day-to-day model using perceived costs as state variables

Different from Horowitz (1984), Smith (1984) proposed the proportional-switch adjustment process (PSAP) (also known as the *Smith* dynamic as in, e.g., Sandholm, 2010). For the same network in Figure 1, the PSAP is formulated as

$$\dot{f}_k = \alpha\left( f_s \max\left\{c_s - c_k, 0\right\} - f_k \max\left\{c_k - c_s, 0\right\}\right),\ \ (k,s) \in \left\{(1,2),(2,1)\right\}$$

PSAP assumes that travelers will switch from higher-cost paths to lower-cost paths, and the switching rate is proportional to the cost difference between each pair of paths and the flow on the higher-cost path. Different from Horowitz (1984)'s model, PSAP uses flows as the state variables, and does not contain any "perceived costs" in its formulation. This initiated another stream of day-to-day models which do not incorporate perceived costs in the model formulation but only flows (and actual travel costs which are functions of flows). A typical structure of this type of models is illustrated in Figure 3: they employ the link/path flows as state variables, and do not include the terms of perceived costs in the model formulation; the dynamics of flows are formulated as functions of flows and actual travel costs, under the assumption that the demands are intended to switch from higher-cost routes to lower-cost routes. Usually, the fixed points of this type of models are Wardrop's UE, and many of them belong to the "rational behavior adjustment process" framework (Guo et al., 2013, 2015; Yang and Zhang, 2009; Zhang et al., 2001), including those in Friesz et al. (1994), Han and Du (2012), He et al. (2010), Jin (2007), Nagurney and Zhang (1997), Smith (1984), and Smith and Mounce (2011). Only a handful of them have other types of fixed points such as boundedly rational UE (Di et al., 2015; Guo and Liu, 2011; Ye and Yang, 2017) or SUE (Guo et al., 2013; Smith and Watling, 2016; Xiao et al., 2019; Yang and Liu, 2007).

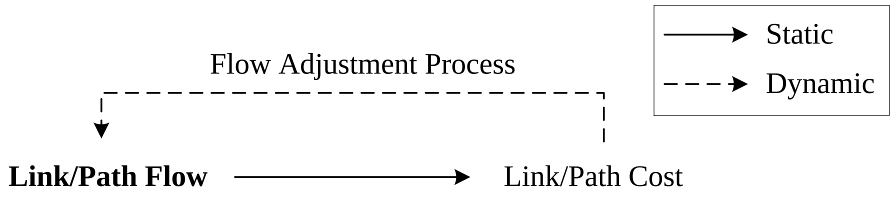


**Figure 3.** Flow chart of a typical day-to-day dynamic using only flows as state variables

Since invented, these two types of day-to-day models (with and without perceived costs in the model formulation) have both achieved extensive development. Although both types of models can simulate the day-to-day evolution of the transportation network towards the same steady states, their inter-relationship is rarely studied. Even though some models do not include the perceived costs in the formulation, it does not mean that travelers do not behave perceiving or learning in these models; in fact, travelers' cognitive process such as perceiving and learning can be one of the underlying factors that cause the observed flow fluctuation in the transportation system and thus determine the flow evolution rules assumed by these models. For instance, Xiao et al. (2016) showed that the double dynamics of flows and perceived costs they proposed is equivalent to a second-order dynamic of flows.

This paper aims at investigating the interconnection between several day-to-day models, some including perceived costs in the model formulation, while others don't. Section 2 focuses on two models. One model has logit-SUE steady states and is a widely used model composed of a dynamic of perceived costs based on exponential smoothing and a static logit SNL (Watling, 1999). It is called the *logit-based exponential-smoothing-and-loading* (logit-ESL) dynamic in this paper. The logit-ESL dynamic is transformed to an equivalent form without perceived costs, which is an extension of the first-in-first-out dynamic in Jin (2007) by adding flow-related terms to the travel costs. Based on this equivalent form, the logit-ESL dynamic is proved to be globally asymptotically stable under nonseparable and monotone cost functions. In Section 3, the day-to-day model formulated by Cantarella and Cascetta (1995) is transformed to an equivalent form without perceived costs, and its global stability under separable travel cost functions is proved. Section 4 introduces other SUE concepts such as extended logit (C-logit and path-size logit) and weibit to the logit-ESL model, derives equivalent forms of these new models, and proves their global stability under different conditions. In Section 5, numerical case studies are conducted to investigate the global stability of several day-to-day models that have been theoretically analyzed in the previous sections, but under more general/common assumptions on the travel cost functions. Section 6 draws the conclusions and proposes possible directions for future research.

## 2. The logit-ESL dynamic and its equivalent form

We consider a general road network with a set $A$ of links and a set $W$ of OD pairs. Each OD pair $w \in W$ has a fixed demand $d_w$ and a set $R_w$ of paths. Let $f_{rw}$ denote the flow on path $r \in R_w$, $w \in W$, $f = \left(f_{rw}, r \in R_w, w \in W\right)^{\mathrm{T}}$ the path flow vector (where "T" represents the transpose operation), and $\Omega = \left\{ f \middle| f > 0, \sum_{r \in R_w} f_{rw} = d_w, w \in W \right\}$ the path flow set where all paths carry positive flow. The path travel cost functions are denoted by $c(f) = \left(c_{rw}(f), r \in R_w, w \in W\right)^{\mathrm{T}}$ where $c_{rw}(f)$ is the cost on path $r \in R_w$, $w \in W$, which is assumed to be continuously differentiable and has a strictly positive lower bound and a finite upper bound.

As reviewed in Section 1, some day-to-day models assume that travelers have their own perceptions on the future travel costs of all paths in the network. The perceived cost includes, as a common assumption for discrete choice models, a deterministic component and a stochastic component. The deterministic component is denoted by $p_{rw}$ for path $r \in R_w$, $w \in W$, and is updated following some learning rules based on the actual travel cost of the same path. The most widely used learning rule is the exponential smoothing rule given below:

$$\dot{p}_{rw} = \eta\left(c_{rw} - p_{rw}\right), \quad \eta > 0, \quad r \in R_w, \quad w \in W \tag{1}$$

We assume that the initial values of all $p_{rw}$ are positive, and therefore $p_{rw}$ are positive along the whole evolution process. The perceived costs then determine the traffic flows via SNL, where most commonly, a logit form is used:

$$f_{rw} = d_w \frac{\exp\left(-\beta p_{rw}\right)}{\sum_{s \in R_w} \exp\left(-\beta p_{sw}\right)}, \quad \beta > 0, \quad r \in R_w, \quad w \in W \tag{2}$$

Eqs. (1) & (2) together describe a day-to-day model which has been widely discussed or used in Watling (1999), Huang et al. (2008), Bie and Lo (2010), Ye and Yang (2013) and Xiao and Lo (2015), just to name a few. In this paper, we call this model the *logit-based exponential-smoothing-and-loading* (logit-ESL) dynamic.

As a key property of the day-to-day models, stability of the logit-ESL dynamic (and other more complex models based on it) has been analyzed only in a local sense (i.e., the local asymptotic stability). In this paper, we will prove that it is also globally stable (in this paper, the global stability refers to the global asymptotic stability, and the rigorous definitions of local and global asymptotic stability are provided in Appendix A). To do so, we first transform the logit-ESL dynamic to an equivalent form which does not include perceived costs, as stated in Theorem 1.

**Theorem 1.** *Model (1) & (2) is equivalent to the dynamic defined by Eq. (3)*:

$$\dot{f}_{rw} = \frac{\beta\eta}{d_w} \sum_{s \in R_w} f_{rw} f_{sw} \left[ c_{sw} - c_{rw} + \frac{1}{\beta} \left( \ln f_{sw} - \ln f_{rw} \right) \right], \quad r \in R_w, \quad w \in W \tag{3}$$

**Proof.** From Eq. (2), we have

$$\frac{f_{rw}}{f_{sw}} = \frac{\exp(-\beta p_{rw})}{\exp(-\beta p_{sw})} \tag{4}$$

and then

$$\ln f_{rw} - \ln f_{sw} = -\beta \left( p_{rw} - p_{sw} \right) \tag{5}$$

Taking derivative on both sides of Eq. (2) with respect to the calendar time yields

$$\begin{aligned} \dot{f}_{rw} &= \beta d_w \frac{\exp(-\beta p_{rw}) \sum_{s \in R_w} \exp(-\beta p_{sw}) \dot{p}_{sw} - \dot{p}_{rw} \exp(-\beta p_{rw}) \sum_{s \in R_w} \exp(-\beta p_{sw})}{\left[ \sum_{s \in R_w} \exp(-\beta p_{sw}) \right]^2} \\ &= \frac{\beta}{d_w} \sum_{s \in R_w} f_{rw} f_{sw} \left( \dot{p}_{sw} - \dot{p}_{rw} \right) \\ &= \frac{\beta}{d_w} \sum_{s \in R_w} f_{rw} f_{sw} \left[ \eta \left( c_{sw} - c_{rw} \right) - \eta \left( p_{sw} - p_{rw} \right) \right] \\ &= \frac{\beta\eta}{d_w} \sum_{s \in R_w} f_{rw} f_{sw} \left[ c_{sw} - c_{rw} + \frac{1}{\beta} \left( \ln f_{sw} - \ln f_{rw} \right) \right] \end{aligned} \tag{6}$$

where the second, third and fourth equalities are obtained by substituting Eqs. (2), (1) and (5), respectively. This completes the proof. □

To verify that the fixed points of dynamic (3) are logit SUE, let $\dot{f}_{rw} = 0$ in Eq. (3), i.e.,

$$\begin{aligned}
&\sum_{s\in R_w} f_{rw} f_{sw} \left[ c_{sw} - c_{rw} + \frac{1}{\beta}\left(\ln f_{sw} - \ln f_{rw}\right) \right] \\
&= f_{rw} \left[ \sum_{s\in R_w} f_{sw} \left( c_{sw} + \frac{1}{\beta}\ln f_{sw} \right) - \sum_{s\in R_w} f_{sw} \left( c_{rw} + \frac{1}{\beta}\ln f_{rw} \right) \right] \\
&= f_{rw} \left[ \sum_{s\in R_w} f_{sw} \left( c_{sw} + \frac{1}{\beta}\ln f_{sw} \right) - d_w \left( c_{rw} + \frac{1}{\beta}\ln f_{rw} \right) \right] \\
&= 0
\end{aligned}$$

In the last equality, since $f_{rw} > 0$, then

$$c_{rw} + \frac{1}{\beta}\ln f_{rw} = \frac{1}{d_w} \sum_{s\in R_w} f_{sw} \left( c_{sw} + \frac{1}{\beta}\ln f_{sw} \right), \;\; \forall r \in R_w, \;\; w \in W$$

and thus

$$c_{rw} + \frac{1}{\beta}\ln f_{rw} = c_{kw} + \frac{1}{\beta}\ln f_{kw}, \;\; \forall r, k \in R_w, \;\; w \in W$$

which implies logit SUE. Therefore, the fixed points of dynamic (3) are logit SUE.

**Remark 1.** In this paper, we call dynamic (3) the *logit-FIFO* dynamic, as its fixed points are logit SUE and it can be treated as an extension of the following first-in-first-out (FIFO) dynamic proposed by Jin (2007),

$$\dot{f}_{rw} = \alpha_w \sum_{s\in R_w} f_{rw} f_{sw} \left( c_{sw} - c_{rw} \right), \;\; \alpha_w > 0, \;\; r \in R_w, \;\; w \in W \tag{7}$$

Mathematically speaking, Eq. (3) is obtained by adding the logarithms of path flows to Eq. (7). From a behavioral point of view, the logit-FIFO dynamic seems to assume that travelers are making decisions based not on the path costs but on the compound costs $c_{rw} + \frac{1}{\beta} \ln f_{rw}$. However, it is not intuitive why $\ln f_{rw}$ will be considered as part of the travel cost. Regarding this, the literature has provided several interpretations.

- Smith and Watling (2016) proposed an extended PSAP model which has logit SUE as the steady states. They assumed that the flows will evolve towards the flow pattern resulted from the logit SNL under current actual costs, and that the flow swapping between two paths is determined by the following odds ratio:

$$\rho_{srw} = \frac{f_{sw} / f_{rw}}{e^{-\beta c_{sw}} / e^{-\beta c_{rw}}}, \quad r, s \in R_w, w \in W \tag{8}$$

which can also be written as

$$\ln \rho_{srw} = \ln f_{sw} - \ln f_{rw} + \beta \left( c_{sw} - c_{rw} \right) \tag{9}$$

Intuitively, if $\rho_{srw} > 1$ (or $\ln \rho_{srw} > 0$), some demand is expected to switch from path $s$ to path $r$, and the opposite if $\rho_{srw} < 1$ (or $\ln \rho_{srw} < 0$). Then a mathematically tractable day-to-day model can be formulated based on the FIFO dynamic as

$$\dot{f}_{rw} = \alpha_w \sum_{s \in R_w} f_{rw} f_{sw} \ln \rho_{srw} = \alpha_w \beta \sum_{s \in R_w} f_{rw} f_{sw} \left[ \frac{1}{\beta} \left( \ln f_{sw} - \ln f_{rw} \right) + c_{sw} - c_{rw} \right] \tag{10}$$

which, when $\alpha_w = \eta / d_w$, is the logit-FIFO dynamic (3). Therefore, following Smith and Watling (2016), the $\ln f_{rw}$ terms are explained as the result of the log odds ratio assumed in the route switching mechanism.

- The term $\ln f_{rw}$ also appears in Chow et al. (2017) where they used a same formulation as the logit-based PSAP in Smith and Watling (2016) to describe the discrete strategy games with uncertainties in the decision making. In the general context of game theory, Chow et al. (2017) interpreted $\ln\left(f_{rw}/d_w\right)$, the logarithm of the probability of choosing a particular strategy (where a strategy is a path in the context of route choice), as the noises which capture the randomness caused by players' irrationality due to making mistakes or taking risks.

- In Xiao et al. (2019), $\ln f_{rw}$ is defined as part of the path potential and analogized with the chemical potential in thermodynamics.
- In this paper, considering the equivalence of the logit-FIFO dynamic (3) and the logit-ESL dynamic (1) & (2), the term $\ln f_{rw}$ actually originates from travelers' random perception errors which follow the Gumbel distribution.

As we can see, via the model transformation in Theorem 1, we can explain the emergence of the logarithmic term $\ln f_{rw}$ in dynamic (3). However, none of the interpretations above can perfectly explain the physical meaning of this term, or say why travelers would consider $\ln f_{rw}$ as part of their travel cost. Although we may guess that this must be related to the Gumbel distribution, it is still unclear how to derive the term directly from the Gumbel distribution, which poses an interesting question for future research. □

**Remark 2.** According to Jin (2007), letting $\bar{c}_w = \frac{1}{d_w}\sum_{s\in R_w} f_{sw}c_{sw}$ denote the average cost between OD pair $w \in W$, we will have

$$\sum_{s\in R_w} f_{sw}\left(c_{sw}-c_{rw}\right) = d_w\left(\bar{c}_w - c_{rw}\right) \tag{11}$$

Substituting Eq. (11) into Eq. (7), the FIFO dynamic can be rewritten as

$$\dot{f}_{rw} = \alpha_w d_w f_{rw}\left(\bar{c}_w - c_{rw}\right) \tag{12}$$

which has the same form of the best-known *replicator dynamic* (Taylor and Jonker, 1978) in evolutionary game theory. This shows an interesting finding that, the replicator dynamic with additive terms $\frac{1}{\beta}\ln f_{rw}$ on cost functions is identical to the logit-ESL dynamic which combines exponential smoothing and logit SNL. □

Theorem 1 shows that the logit-ESL dynamic (which assumes the perception and learning behavior and incorporates the terms of perceived costs in the model formulation) is equivalent to the logit-FIFO dynamic (whose formulation does not include the perceived costs but only the path flows and actual travel costs). This equivalence indicates that, although the logit-FIFO dynamic does not explicitly assume travelers' perceiving and learning process, it may indeed reflect such behavior. This is similar to the case in Xiao et al. (2016) which showed that a double dynamic of flows and perceived costs is equivalent to a dynamic based purely on flows. Moreover, such equivalence brings two conveniences. On one hand, although explicitly modeling the cognitive process (e.g., perceiving and learning) can help build a more practical day-to-day model, its calibration is difficult as cognition is difficult to measure or estimate. With the equivalent form, the cognition-related parameters $\beta$ and $\eta$ in model (1) & (2) can be calibrated via form (3) using the path flow and travel cost data, which may be easier to collect in practice, either from lab experiments (Ye et al., 2018) or from real urban road networks (via GPS, Bluetooth and other sensors, license plate recognition, etc.). On the other hand, the formulation of the logit-FIFO dynamic facilitates the analysis on the global stability of the logit-ESL model (with nonseparable travel cost functions), as given by the following theorem.

**Theorem 2.** *If the path travel cost functions are monotone, i.e.,*

$$(f-g)^{\mathrm{T}}\left[c(f)-c(g)\right]\geq 0,\ \ \forall f,g\in\Omega \tag{13}$$

*then under the logit-FIFO dynamic (3), any path flow pattern in* $\Omega$ *will converge to the logit SUE.*

**Remark 3.** The monotonicity condition (13) is satisfied if the link travel cost functions are monotone and additive (being additive means $c_{rw}=\sum_{a\in A}\delta_{ar}c_a$, where $c_a$ is the cost on link $a\in A$, $\delta_{ar}=1$ if path $r$ uses link $a$, and $\delta_{ar}=0$ otherwise). □

To prove Theorem 2, we need the following lemma.

**Lemma 1.** *There exists a real number* $Y>0$, *such that given any* $Y'\in(0,Y]$, *the compact (closed and bounded) set*

$$\tilde{\Omega}(Y')=\left\{f\,\middle|\,f\in\Omega,\,f\geq Y'\right\} \tag{14}$$

*is a positively invariant set with respect to dynamic (3), where* $f\geq Y'$ *means* $f_{rw}\geq Y'$ *for all* $r\in R_w$, $w\in W$, *and the definition of (positively) invariant set is given in Appendix B.*

**Proof of Lemma 1.** Since the link costs have a strictly positive lower bound and a finite upper bound, then so do the path costs. Thus we can define

$$Q \triangleq \inf \left\{ c_{sw}(f) - c_{rw}(f) \,|\, s, r \in R_w, w \in W, f \in \Omega \right\}$$

to be finite, where "inf" stands for infimum. Therefore, for the right-hand side of Eq. (3),

$$\begin{aligned}
&\sum_{s \in R_w} f_{rw} f_{sw} \left[ c_{sw} - c_{rw} + \frac{1}{\beta} (\ln f_{sw} - \ln f_{rw}) \right] \\
&\geq f_{rw} \sum_{s \in R_w} f_{sw} \left[ Q + \frac{1}{\beta} (\ln f_{sw} - \ln f_{rw}) \right] \\
&= f_{rw} \left( \sum_{s \in R_w} f_{sw} Q + \frac{1}{\beta} \sum_{s \in R_w} f_{sw} \ln f_{sw} - \frac{1}{\beta} \sum_{s \in R_w} f_{sw} \ln f_{rw} \right) \\
&= f_{rw} \left( Q d_w + \frac{1}{\beta} \sum_{s \in R_w} f_{sw} \ln f_{sw} - \frac{1}{\beta} d_w \ln f_{rw} \right) \\
&\geq f_{rw} \left( Q d_w + \frac{1}{\beta} \inf_{f \in \Omega} \sum_{s \in R_w} f_{sw} \ln f_{sw} - \frac{1}{\beta} d_w \ln f_{rw} \right)
\end{aligned} \tag{15}$$

where the last inequality holds because $\lim_{f_{sw} \to 0+} f_{sw} \ln f_{sw} = 0$ and thus $\inf_{f \in \Omega} \left( \sum_{s \in R_w} f_{sw} \ln f_{sw} \right)$ exists and is finite. Substituting Eq. (15) into Eq. (3), we have $\dot{f}_{rw} \geq 0$ if

$$Q d_w + \frac{1}{\beta} \inf_{f \in \Omega} \sum_{s \in R_w} f_{sw} \ln f_{sw} - \frac{1}{\beta} d_w \ln f_{rw} \geq 0 \tag{16}$$

i.e.,

$$f_{rw} \leq \exp\left( \beta Q + \frac{1}{d_w} \inf_{f \in \Omega} \sum_{s \in R_w} f_{sw} \ln f_{sw} \right) \tag{17}$$

Further, let

$$Y \triangleq \min_{w \in W} \exp\left( \beta Q + \frac{1}{d_w} \inf_{f \in \Omega} \sum_{s \in R_w} f_{sw} \ln f_{sw} \right) > 0 \tag{18}$$

then combining Eqs. (3) and (15)-(18), we have that, if $f_{rw} \leq Y$, then $\dot{f}_{rw} \geq 0$. Therefore, referring to Definition B1 of Appendix B, the set $\tilde{\Omega}(Y')$ defined in (14) is a positively invariant set of dynamic (3). This completes the proof. □

Now we are ready to prove Theorem 2.

**Proof of Theorem 2.** Let

$$\tilde{c}(f)=c(f)+\frac{1}{\beta}\ln f$$

then the logit-FIFO dynamic (3) is rewritten as

$$\dot{f}_{rw}=\frac{\beta\eta}{d_w}\sum_{s\in R_w} f_{rw} f_{sw}\left(\tilde{c}_{sw}-\tilde{c}_{rw}\right) \tag{19}$$

Assume $f^*=\left(f_{rw}^*, r\in R_w, w\in W\right)^{\mathrm{T}}$ to be a fixed point of dynamic (3) and thus a logit SUE, then it satisfies the following variational inequality (Zhou et al., 2012),

$$\left(f-f^*\right)^{\mathrm{T}}\tilde{c}\left(f^*\right)\geq 0,\ \ \forall f\in\Omega \tag{20}$$

Further, if $c(f)$ satisfies the monotonicity condition (13), then $\tilde{c}(f)$ is strictly monotone, i.e.,

$$(f-g)^{\mathrm{T}}\left[\tilde{c}(f)-\tilde{c}(g)\right]>0,\ \ \forall f,g\in\Omega,\ \ f\neq g \tag{21}$$

as the $\ln(\cdot)$ function is strictly increasing. In this case, $f^*$ is the unique logit SUE (Theorem 1.6, Nagurney, 1999) and thus the unique fixed point of dynamic (3). To prove that $f^*$ is globally stable, we adopt the following Lyapunov function which was used by Hofbauer and Sandholm (2009) for the replicator dynamic,

$$V_1(f)=-\frac{1}{\beta\eta}\sum_{w\in W}\sum_{r\in R_w} f_{rw}^*\ln f_{rw} \tag{22}$$

The derivative of $V_1(f)$ with respect to calendar time reads

$$
\begin{aligned}
\dot{V}_1(f) &= -\frac{1}{\beta\eta}\sum_{w\in W}\sum_{r\in R_w} f_{rw}^* \frac{\dot{f}_{rw}}{f_{rw}} \\
&= -\frac{1}{\beta\eta}\sum_{w\in W}\sum_{r\in R_w} f_{rw}^* \frac{1}{f_{rw}}\frac{\beta\eta}{d_w}\sum_{s\in R_w} f_{rw} f_{sw}\left(\tilde{c}_{sw}-\tilde{c}_{rw}\right) \\
&= -\sum_{w\in W}\frac{1}{d_w}\sum_{r\in R_w}\sum_{s\in R_w} f_{rw}^* f_{sw}\left(\tilde{c}_{sw}-\tilde{c}_{rw}\right) \\
&= -\sum_{w\in W}\frac{1}{d_w}\sum_{r\in R_w}\sum_{s\in R_w} f_{rw}^* f_{sw}\tilde{c}_{sw} + \sum_{w\in W}\frac{1}{d_w}\sum_{r\in R_w}\sum_{s\in R_w} f_{rw}^* f_{sw}\tilde{c}_{rw} \\
&= -\sum_{w\in W}\sum_{s\in R_w} f_{sw}\tilde{c}_{sw} + \sum_{w\in W}\sum_{r\in R_w} f_{rw}^*\tilde{c}_{rw} \\
&= \sum_{w\in W}\sum_{r\in R_w}\left(f_{rw}^* - f_{rw}\right)\tilde{c}_{rw} \\
&= \left(f^* - f\right)^{\mathrm{T}}\tilde{c}(f) \\
&\le \left(f^* - f\right)^{\mathrm{T}}\tilde{c}\left(f^*\right) \\
&\le 0
\end{aligned}
$$

where the first and second inequalities hold due to Eqs. (21) and (20), respectively, and $\dot{V}_1(f)=0$ iff $f = f^*$, i.e., $\left\{f \middle| \dot{V}_1(f)=0, f\in\Omega\right\}=\left\{f^*\right\}$. Obviously, the largest invariant set in $\left\{f^*\right\}$ is itself.

Further, according to Lemma 1, the compact set $\tilde{\Omega}(Y')$ defined in (14) is a positively invariant set of dynamic (3). Since the right-hand side of Eq. (3) is differentiable in $\Omega$ and thus Lipschitz continuous, then by LaSalle's theorem (Theorem B1 in Appendix B), if $f^*\in\tilde{\Omega}(Y')$, then any path flow pattern in $\tilde{\Omega}(Y')$ will converge to $f^*$. As in definition (14) of $\tilde{\Omega}(Y')$, $Y'$ can be arbitrarily close to zero, and $\lim_{Y'\to 0}\tilde{\Omega}(Y')=\Omega$, then it actually means that, any flow pattern in $\Omega$ will converge to $f^*$. This completes the proof. □

## 3. Logit-ESL dynamic with logit-dynamic flow adjustment

Cantarella and Cascetta (1995) proposed a day-to-day model considering the dynamics of both flows and perceived costs. The perceived costs are updated following the same rule as Eq. (1), and the flows are assumed to evolve following dynamic (23):

$$\dot{f}_{rw} = \alpha \left[ d_w \frac{\exp(-\beta p_{rw})}{\sum_{s \in R_w} \exp(-\beta p_{sw})} - f_{rw} \right], \quad \alpha > 0, \quad r \in R_w, \quad w \in W \tag{23}$$

Model (23) & (1) can be treated as an extension of the logit-ESL model (1) & (2) when the path flows follow a dynamical adjustment process (23) rather than the static SNL (2). It can also be treated as an extension of dynamic (24), the so-called logit dynamic in the transportation context (Yang, 2005) or the perturbed best response dynamic with logit choice rule in evolutionary game theory (Fudenberg and Levine, 1998):

$$\dot{f}_{rw} = \alpha \left[ d_w \frac{\exp(-\beta c_{rw}(f))}{\sum_{s \in R_w} \exp(-\beta c_{sw}(f))} - f_{rw} \right] \tag{24}$$

Similar to transforming the logit-ESL dynamic in Theorem 1, Cantarella and Cascetta (1995)'s model (23) & (1) can also be transformed to a form with no perceived costs. Let $x_{rw} = \frac{1}{\alpha} \dot{f}_{rw} + f_{rw}$, then Eq. (23) can be rewritten as

$$x_{rw} = d_w \frac{\exp(-\beta p_{rw})}{\sum_{s \in R_w} \exp(-\beta p_{sw})} \tag{25}$$

Further let $x = (x_{rw}, r \in R_w, w \in W)^{\mathrm{T}}$, then the feasible region of $x$ is also $\Omega$. Taking derivative on both sides of Eq. (25) and following the procedure in the proof of Theorem 1, we can transform model (23) & (1) into the following equivalent form (26) & (27).

$$\dot{f}_{rw} = \alpha (x_{rw} - f_{rw}) \tag{26}$$

$$\dot{x}_{rw} = \frac{\beta\eta}{d_w} \sum_{s \in R_w} x_{rw} x_{sw} \left[ c_{sw}(f) - c_{rw}(f) + \frac{1}{\beta} (\ln x_{sw} - \ln x_{rw}) \right] \tag{27}$$

Note that model (26) & (27) is a second-order dynamic which includes the first- and second-order dynamics of just path flows but no perceived costs. The equivalence between model (23) & (1) (which includes the dynamics of both flows and perceived costs) and model (26) & (27) (which includes the dynamics of only flows) is similar to that in Xiao et al. (2016) which found that, the double dynamics of flows and perceived costs that they formulated is equivalent to a second-order dynamic of flows. However, Xiao et al. (2016)'s model is based on Wardrop's UE, while Cantarella and Cascetta (1995)'s model (23) & (1) is based on SUE.

With the help of the equivalent model (26) & (27) established above, we can prove that Cantarella and Cascetta (1995)'s model (23) & (1) is globally stable under separable link travel cost functions, in Theorem 3 below.

**Theorem 3.** *If the link travel cost functions are separable and additive, then under dynamic (23) & (1), when the path flow pattern is in* $\Omega$ *and the perceived costs are positive, then they will converge to the logit SUE path flows and path costs, respectively.*

**Proof.** To prove the theorem, we first prove that dynamic (26) & (27), i.e., the equivalent form of dynamic (23) & (1), is globally stable.

First, comparing Eqs. (27) and (3), we have that, similar to Lemma 1, given any $Y' \in (0, Y]$, where $Y$ is defined in Eq. (18), the compact set $\tilde{\Omega}(Y')$ defined in (14) is also a positively invariant set of $x$. Further, according to Eq. (26), if $x_{rw} \geq Y'$ and $f_{rw} \leq Y'$, then $x_{rw} - f_{rw} \geq 0$ and thus $\dot{f}_{rw} \geq 0$. This means, given $Y' \in (0, Y]$, $x \in \tilde{\Omega}(Y')$ and any $Y'' \in (0, Y']$, the compact set

$$\tilde{\Omega}(Y'') = \{ f \mid f \in \Omega, f \geq Y'' \} \tag{28}$$

will be a positively invariant set of $f$. For convenience, we define the set

$$G(Y', Y'') = \{ (f, x) \mid Y' \in (0, Y], Y'' \in (0, Y'], x \in \tilde{\Omega}(Y'), f \in \tilde{\Omega}(Y'') \} \tag{29}$$

which is a positively invariant set of $(f, x)$ with respect to dynamic (26) & (27).

Now, define

$$V_2(f, x) = -\alpha \sum_{w \in W} \sum_{r \in R_w} f_{rw} \ln x_{rw} + (\alpha + \eta) \sum_{w \in W} \sum_{r \in R_w} f_{rw} (\ln f_{rw} - 1) + \beta\eta \sum_{a \in A} \int_0^{v_a(f)} c_a(\omega) d\omega$$

where $v_a(f)$ is the flow on link $a \in A$ given path flow pattern $f$. Note that, since $x_{rw} \leq d_w$ (according to Eq. (25)) and $f \in \Omega$, then

$$-\alpha \sum_{w \in W} \sum_{r \in R_w} f_{rw} \ln x_{rw} \geq -\alpha \sum_{w \in W} \sum_{r \in R_w} f_{rw} \ln d_w = -\alpha \sum_{w \in W} d_w \ln d_w$$

and $\inf_{f \in \Omega} \left( \sum_{r \in R_w} f_{rw} \ln f_{rw} \right)$ is finite (as $\lim_{f_{rw} \to 0+} f_{rw} \ln f_{rw} = 0$). Therefore, $V_2(f, x)$ is bounded from below by a finite number.

Since the link travel cost functions are assumed to be separable and additive, then

$$\frac{\partial V_2(f, x)}{\partial f_{rw}} = -\alpha \ln x_{rw} + (\alpha + \eta) \ln f_{rw} + \beta\eta c_{rw} \tag{30}$$

and

$$\frac{\partial V_2(f,x)}{\partial x_{rw}} = -\alpha \frac{f_{rw}}{x_{rw}} \tag{31}$$

The derivative of $V_2(f,x)$ with respect to calendar time is then calculated as

$$\dot{V}_2(f,x) = \sum_{w\in W}\sum_{r\in R_w} \frac{\partial V_2(f,x)}{\partial f_{rw}} \dot{f}_{rw} + \sum_{w\in W}\sum_{r\in R_w} \frac{\partial V_2(f,x)}{\partial x_{rw}} \dot{x}_{rw} \tag{32}$$

In Eq. (32), the second term on the right-hand side reads

$$\begin{aligned}
&\sum_{w\in W}\sum_{r\in R_w} \frac{\partial V_2(f,x)}{\partial x_{rw}} \dot{x}_{rw} \\
&= -\sum_{w\in W}\sum_{r\in R_w} \alpha \frac{f_{rw}}{x_{rw}} \frac{\beta\eta}{d_w} \sum_{s\in R_w} x_{rw} x_{sw} \left[ c_{sw} - c_{rw} + \frac{1}{\beta}\left(\ln x_{sw} - \ln x_{rw}\right)\right] \\
&= -\alpha\beta\eta \sum_{w\in W} \frac{1}{d_w} \left[ \sum_{r\in R_w} f_{rw} \sum_{s\in R_w} x_{sw} \left( c_{sw} + \frac{1}{\beta}\ln x_{sw}\right) - \sum_{r\in R_w} f_{rw} \sum_{s\in R_w} x_{sw} \left( c_{rw} + \frac{1}{\beta}\ln x_{rw}\right)\right] \\
&= -\alpha\beta\eta \sum_{w\in W} \left[ \sum_{s\in R_w} x_{sw} \left( c_{sw} + \frac{1}{\beta}\ln x_{sw}\right) - \sum_{r\in R_w} f_{rw} \left( c_{rw} + \frac{1}{\beta}\ln x_{rw}\right)\right] \\
&= -\alpha\beta\eta \sum_{w\in W}\sum_{r\in R_w} \left(x_{rw} - f_{rw}\right)\left( c_{rw} + \frac{1}{\beta}\ln x_{rw}\right)
\end{aligned} \tag{33}$$

where the first equality is obtained by substituting Eqs. (27) and (31), and the third equality holds because $\sum_{r\in R_w} f_{rw} = \sum_{s\in R_w} x_{sw} = d_w$. Substituting Eqs. (33), (30) and (26) into Eq. (32), we further get

$$\begin{aligned}
\dot{V}_2(f,x) &= \sum_{w\in W}\sum_{r\in R_w} \left[ -\alpha \ln x_{rw} + (\alpha+\eta)\ln f_{rw} + \beta\eta c_{rw} \right] \alpha \left( x_{rw} - f_{rw}\right) \\
&\quad - \alpha\beta\eta \sum_{w\in W}\sum_{r\in R_w} \left(x_{rw} - f_{rw}\right)\left( c_{rw} + \frac{1}{\beta}\ln x_{rw}\right) \\
&= -\alpha(\alpha+\eta) \sum_{w\in W}\sum_{r\in R_w} \left(\ln x_{rw} - \ln f_{rw}\right)\left(x_{rw} - f_{rw}\right) \\
&\le 0
\end{aligned}$$

where the inequality holds because $\ln(\cdot)$ is strictly increasing. Hence $\dot{V}_2(f,x) = 0$ iff $f = x$.

To show the convergence of $f$ and $x$, let

$$\Psi \triangleq \left\{ (f, x) \middle| \dot{V}_2(f, x) = 0, f \in \Omega, x \in \Omega \right\} = \left\{ (f, x) \middle| f = x, f \in \Omega, x \in \Omega \right\}$$

Since the right-hand side of dynamic (26) & (27) is differentiable and thus Lipschitz continuous, then by LaSalle's theorem, under dynamic (26) & (27), if $(f, x) \in G(Y', Y'')$ (where $G(Y', Y'')$ is the positively invariant set defined in Eq. (29)) and $G(Y', Y'') \bigcap \Psi \neq \varnothing$, then the pattern $(f, x)$ will converge to the largest invariant set in $G(Y', Y'') \bigcap \Psi$. Further, since in the definition of $G(Y', Y'')$, $Y'$ and $Y''$ can be arbitrarily close to zero, and $\lim_{Y' \to 0, Y'' \to 0} \left( G(Y', Y'') \bigcap \Psi \right) = \Psi$, then any $(f, x)$ pattern satisfying $f \in \Omega$ and $x \in \Omega$ will converge to the largest invariant set in $\Psi$.

To find out the largest invariant set in $\Psi$, suppose at time $t_0$, $\left( f(t_0), x(t_0) \right) \in \Psi$, i.e., $f(t_0) = x(t_0) \triangleq f^{\#} \triangleq \left( f_{rw}^{\#}, r \in R_w, w \in W \right)^{\mathrm{T}}$. Letting $\left( f(t), x(t) \right) \in \Psi$ for all $t \geq t_0$, we have

$$f(t) \equiv x(t), \quad \forall t \geq t_0 \tag{34}$$

Substituting condition (34) into Eq. (26) leads to $\dot{f}(t) \equiv 0$ and thus $f(t) \equiv f(t_0) = f^{\#}$ for all $t \geq t_0$. Combining this with condition (34), we have

$$x(t) \equiv f^{\#}, \quad \forall\ t \geq t_0 \tag{35}$$

Substituting condition (35) into Eq. (27) gives us

$$\dot{x}_{rw} = \frac{\beta \eta}{d_w} \sum_{s \in R_w} f_{rw}^{\#} f_{sw}^{\#} \left[ c_{sw}\left( f^{\#} \right) - c_{rw}\left( f^{\#} \right) + \frac{1}{\beta} \left( \ln f_{sw}^{\#} - \ln f_{rw}^{\#} \right) \right] = 0, \quad r \in R_w, \quad w \in W \tag{36}$$

According to Section 2, condition (36) holds iff $f^{\#}$ is logit SUE. This means the largest invariant set in $\Psi$ is $\left\{ (f, x) \middle| f = x = \text{logit SUE} \right\}$. Therefore, any flow pattern $f \in \Omega$ will converge to the logit SUE path flow pattern. Consequently, according to Eq. (1), the perceived costs $p_{rw}$ will converge to the logit SUE path costs. This completes the proof. □

## 4. Day-to-day dynamics based on other SUE concepts

The logit model is popular due to its closed-form formulation and thus the convenience for analysis and calibration. However, it is deficient in dealing with overlapping paths and heterogeneous perception errors. To overcome these shortcomings while maintaining a closed-form expression of choice probabilities, various discrete choice models have been proposed. Some models modify the deterministic terms in the utility functions to consider the impact of path overlapping while maintaining the logit structure. These include the C-logit (Cascetta et al., 1996), the path-size logit (PS-logit) (Ben-Akiva and Bierlaire, 1999) and so on. Some models change the assumptions on the error terms to follow the Weibull distribution (Castillo et al., 2008) or to be multiplicative (Fosgerau and Bierlaire, 2009), leading to the weibit model. In this section, we use these alternative discrete choice models to formulate new day-to-day dynamics.

### *4.1. C-logit and PS-logit*

The C-logit and PS-logit extend the standard logit model by adding path-specific constants to the path costs, and can be described by the following unified form:

$$f_{rw} = d_w \frac{\exp\left[-\beta\left(p_{rw} + \gamma_{rw}\right)\right]}{\sum_{s\in R_w} \exp\left[-\beta\left(p_{sw} + \gamma_{sw}\right)\right]}, \quad r \in R_w, \quad w \in W \tag{37}$$

where $\gamma_{rw}$ are path-specific constants with different interpretations in C-logit and PS-logit (Chen et al., 2012). Eqs. (37) & (1) then give a day-to-day model which is more general than the logit-ESL dynamic (1) & (2). Following the same procedure in the proof of Theorem 1, model (37) & (1) can be transformed to the following logit-FIFO-like form (named C/PS-logit-FIFO):

$$\dot{f}_{rw} = \frac{\beta\eta}{d_w} \sum_{s \in R_w} f_{rw} f_{sw} \left[ (c_{sw} + \gamma_{sw}) - (c_{rw} + \gamma_{rw}) + \frac{1}{\beta} (\ln f_{sw} - \ln f_{rw}) \right] \tag{38}$$

which can be proved to be globally stable under the monotonicity condition (13), by following the same proof for Theorem 2 (the complete proof is omitted here).

### 4.2. Weibit

Based on the weibit model (Castillo et al., 2008; Fosgerau and Bierlaire, 2009), the weibit-SUE path flow pattern satisfies

$$f_{rw} = d_w \frac{(c_{rw}(f))^{-\beta}}{\sum_{s \in R_w} (c_{sw}(f))^{-\beta}}, \quad \beta > 0, \quad r \in R_w, \quad w \in W \tag{39}$$

As the path costs are assumed to be positive, the path flows are always positive and their feasible region is also $\Omega$.

Using the same technique of creating the logit-FIFO dynamic via the log odds ratio in Eq. (10), we can get the following weibit-FIFO dynamic:

$$\dot{f}_{rw} = \alpha_w \sum_{s \in R_w} f_{rw} f_{sw} \ln \frac{f_{sw} / f_{rw}}{c_{sw}^{-\beta} / c_{rw}^{-\beta}} = \alpha_w \beta \sum_{s \in R_w} f_{rw} f_{sw} \left[ (\ln c_{sw} - \ln c_{rw}) + \frac{1}{\beta} (\ln f_{sw} - \ln f_{rw}) \right] \tag{40}$$

Comparing the forms of the logit-FIFO dynamic (10) and the weibit-FIFO dynamic (40), their only difference lies in that, the travel costs in logit-FIFO dynamic are replaced by logarithms of travel costs in weibit-FIFO dynamic. This is because, as pointed out by Castillo et al. (2008) and Fosgerau and Bierlaire (2009), the weibit model (39) can also be treated as a logit model with respect to the logarithms of path costs, as shown below:

$$f_{rw} = d_w \frac{\exp\left(-\beta \ln c_{rw}\left(f\right)\right)}{\sum_{s\in R_w} \exp\left(-\beta \ln c_{sw}\left(f\right)\right)} \tag{41}$$

Recalling the equivalence of logit-ESL dynamic and logit-FIFO dynamic, a natural question is, whether the weibit-FIFO dynamic also has an equivalent form which reflects travelers' perceiving and learning behavior. Below, we provide two weibit-based models which consider such behavior, named the type-I and type-II weibit-ESL dynamic, respectively, and we will show that only the type-I dynamic is equivalent to the weibit-FIFO dynamic.

**Type-I weibit-ESL dynamic.** Referring to Xu et al. (2015), the alternative formulation (41) of the weibit model can be interpreted as, for long trips, travelers may evaluate their travel costs in a logarithmic manner. Therefore, we can assume travelers' learning rule to be

$$\dot{p}_{rw} = \eta\left(\ln c_{rw} - p_{rw}\right),\ \ \eta > 0,\ \ r \in R_w,\ \ w \in W \tag{42}$$

while using the same logit SNL in Eq. (2) to generate the path flows. We name dynamic (42) & (2) the *type-I weibit-ESL* dynamic as its steady states are weibit SUE. This dynamic is equivalent to the weibit-FIFO dynamic as revealed by the following theorem.

**Theorem 4.** *The type-I weibit-ESL dynamic (42) & (2) is equivalent to the weibit-FIFO dynamic (40).*

**Proof.** Following the same procedure in the proof of Theorem 1 while using the new learning rule in Eq. (42), we readily have

$$\dot{f}_{rw} = \frac{\beta\eta}{d_w} \sum_{s \in R_w} f_{rw} f_{sw} \left[ \left( \ln c_{sw} - \ln c_{rw} \right) + \frac{1}{\beta} \left( \ln f_{sw} - \ln f_{rw} \right) \right] \tag{43}$$

which has the same form as the weibit-FIFO dynamic in Eq. (40). The proof is thus completed. □

**Type-II weibit-ESL dynamic.** To build our second weibit-based day-to-day model, we can use the same learning rule in Eq. (1), and adopt weibit SNL to get the path flows as

$$f_{rw} = d_w \frac{p_{rw}^{-\beta}}{\sum_{s \in R_w} p_{sw}^{-\beta}} \tag{44}$$

Eqs. (44) and (1) then give us the *type-II weibit-ESL* dynamic. Taking derivative on both sides of Eq. (44) and applying the similar procedure in Eq. (6), we get

$$\begin{aligned}
\dot{f}_{rw} &= d_w \frac{-\beta p_{rw}^{-\beta-1} \dot{p}_{rw} \sum_{s \in R_w} p_{sw}^{-\beta} - p_{rw}^{-\beta} \sum_{s \in R_w} (-\beta) p_{sw}^{-\beta-1} \dot{p}_{sw}}{\left( \sum_{s \in R_w} p_{sw}^{-\beta} \right)^2} \\
&= -\beta \left[ d_w \frac{p_{rw}^{-\beta-1} \dot{p}_{rw} \sum_{s \in R_w} p_{sw}^{-\beta}}{\left( \sum_{s \in R_w} p_{sw}^{-\beta} \right)^2} - d_w \frac{p_{rw}^{-\beta} \sum_{s \in R_w} p_{sw}^{-\beta-1} \dot{p}_{sw}}{\left( \sum_{s \in R_w} p_{sw}^{-\beta} \right)^2} \right] \\
&= -\beta \left( \frac{f_{rw}}{p_{rw}} \dot{p}_{rw} - f_{rw} \sum_{s \in R_w} \frac{1}{d_w} \frac{f_{sw}}{p_{sw}} \dot{p}_{sw} \right) \\
&= \frac{\beta}{d_w} \sum_{s \in R_w} f_{rw} f_{sw} \left( \frac{\dot{p}_{sw}}{p_{sw}} - \frac{\dot{p}_{rw}}{p_{rw}} \right) \\
&= \frac{\beta}{d_w} \sum_{s \in R_w} f_{rw} f_{sw} \left[ \frac{\eta \left( c_{sw} - p_{sw} \right)}{p_{sw}} - \frac{\eta \left( c_{rw} - p_{rw} \right)}{p_{rw}} \right] \\
&= \frac{\beta\eta}{d_w} \sum_{s \in R_w} f_{rw} f_{sw} \left( \frac{c_{sw}}{p_{sw}} - \frac{c_{rw}}{p_{rw}} \right)
\end{aligned} \tag{45}$$

where the third and fifth equalities are obtained by substituting Eqs. (44) and (1), respectively. Further, according to Eq. (44), given a particular path $k \in R_w$ between each OD pair

$w \in W$, we have

$$p_{rw} = \left( \frac{f_{rw}}{f_{kw}} \right)^{-\frac{1}{\beta}} p_{kw}, \quad p_{sw} = \left( \frac{f_{sw}}{f_{kw}} \right)^{-\frac{1}{\beta}} p_{kw} \tag{46}$$

Substituting Eq. (46) into Eq. (45) yields

$$\dot{f}_{rw} = \frac{\beta\eta}{d_w} \frac{1}{p_{kw} f_{kw}^{1/\beta}} \sum_{s \in R_w} f_{rw} f_{sw} \left( c_{sw} f_{sw}^{1/\beta} - c_{rw} f_{rw}^{1/\beta} \right) \tag{47}$$

The form (47) still includes the perceived cost of path $k$, and thus needs to be combined with Eq. (1) to describe a complete day-to-day model. Therefore, the type-II weibit-ESL dynamic (44) & (1) is not equivalent to the weibit-FIFO dynamic (40).

The equivalent forms of the type-I and type-II weibit-ESL dynamics facilitate the analysis on their global stability, which is provided by the subsequent Theorem 5 and Theorem 6. The two theorems require the path travel cost functions to possess different monotonicity properties, which will be specified in each theorem.

**Theorem 5.** *If the logarithms of path travel cost functions are monotone, i.e.,*

$$(f - g)^{\mathrm{T}} \left[ \ln c(f) - \ln c(g) \right] \geq 0, \quad \forall f, g \in \Omega \tag{48}$$

*then under the type-I weibit-ESL dynamic (42) & (2), or equivalently the weibit-FIFO dynamic (40), any path flow pattern in* $\Omega$ *will converge to the weibit SUE.*

**Remark 4.** For simplicity, hereafter we call a function $c(\cdot)$ satisfying condition (48) to be *log-monotone*. Generally speaking, $c(\cdot)$ being monotone does not mean that it is log-monotone, and vice versa. However, condition (48) is satisfied if the link travel cost functions are log-monotone and multiplicative. The link travel cost functions would be log-monotone if they are separable and increasing. The link costs being multiplicative means $c_{rw} = \prod_{a \in A} c_a^{\delta_{ar}}$ , or equivalently the logarithmic link costs being additive, i.e., $\ln c_{rw} = \sum_{a \in A} \delta_{ar} \ln c_a$ , which was the assumption adopted in Kitthamkesorn and Chen (2013, 2014) when solving the weibit SUE via mathematical programming. □

**Proof of Theorem 5.** Referring to the alternative formulation (41) of the weibit SUE and the monotonicity of $\ln c(f)$ in condition (48), Theorem 5 can be readily proved by replacing $c(f)$ in the proofs of Lemma 1 and Theorem 2 with $\ln c(f)$. Just note that, since the path costs have a strictly positive lower bound, then the logarithms of path costs would have a finite lower bound. The complete proof is omitted here. □

For the type-II weibit-ESL dynamic (1) & (44), its global stability can be proved when the link travel cost functions are separable, which is stated below.

**Theorem 6.** *If the link travel cost functions are multiplicative and separable, then under the type-II weibit-ESL dynamic (1) & (44), or equivalently the dynamic (1) & (47), any feasible path flows and positive perceived costs will converge to the weibit-SUE path flows and path costs, respectively.*

To prove Theorem 6, we need the following lemma.

**Lemma 2.** *Under the type-II weibit-ESL dynamic (1) & (47), there exists a number* $L > 0$, *such that given any* $L' \in (0, L]$, *the compact set*

$$\tilde{\Omega}(L') = \{ f \mid f \in \Omega, f \geq L' \} \tag{49}$$

*is a positively invariant set of the path flows.*

**Proof of Lemma 2.** The proof is similar to the proof of Lemma 1.

Since the lower and upper bounds of the path travel costs are finite and positive, we can define

$$Q_{\inf} \triangleq \inf \{ c_{rw}(f) \mid r \in R_w, w \in W, f \in \Omega \} > 0 \tag{50}$$

and

$$Q_{\sup} \triangleq \sup \{ c_{rw}(f) \mid r \in R_w, w \in W, f \in \Omega \} > 0$$

where "sup" stands for supremum. Therefore, for the right-hand side of Eq. (47),

$$\begin{aligned} \sum_{s \in R_w} f_{rw} f_{sw} \left( c_{sw} f_{sw}^{1/\beta} - c_{rw} f_{rw}^{1/\beta} \right) &\geq \sum_{s \in R_w} f_{rw} f_{sw} \left( Q_{\inf} f_{sw}^{1/\beta} - Q_{\sup} f_{rw}^{1/\beta} \right) \\ &= f_{rw} \left( Q_{\inf} \sum_{s \in R_w} f_{sw}^{(\beta+1)/\beta} - Q_{\sup} d_w f_{rw}^{1/\beta} \right) \\ &\geq f_{rw} \left( Q_{\inf} \inf_{f \in \Omega} \sum_{s \in R_w} f_{sw}^{(\beta+1)/\beta} - Q_{\sup} d_w f_{rw}^{1/\beta} \right) \end{aligned} \tag{51}$$

Substituting Eq. (51) into Eq. (47), we have

$$\dot{f}_{rw} \geq \frac{\beta\eta}{d_w} \frac{1}{p_{kw} f_{kw}^{1/\beta}} f_{rw} \left( Q_{\inf} \inf_{f \in \Omega} \sum_{s \in R_w} f_{sw}^{(\beta+1)/\beta} - Q_{\sup} d_w f_{rw}^{1/\beta} \right) \tag{52}$$

Since $p_{kw}$ is always positive (according to Eq. (1)), then from Eq. (52), we have $\dot{f}_{rw} \geq 0$ if

$$Q_{\inf} \inf_{f \in \Omega} \sum_{s \in R_w} f_{sw}^{(\beta+1)/\beta} - Q_{\sup} d_w f_{rw}^{1/\beta} \geq 0$$

i.e.,

$$f_{rw} \leq \left[ \frac{Q_{\inf}}{Q_{\sup} d_w} \inf_{f \in \Omega} \sum_{s \in R_w} f_{sw}^{(\beta+1)/\beta} \right]^{\beta} \tag{53}$$

Note that Eq. (53) is valid because $\inf_{f \in \Omega} \sum_{s \in R_w} f_{sw}^{(\beta+1)/\beta}$ is strictly positive due to the constraint $f \in \Omega$. Define

$$L = \min_{w \in W} \left[ \frac{Q_{\inf}}{Q_{\sup} d_w} \inf_{f \in \Omega} \sum_{s \in R_w} f_{sw}^{(\beta+1)/\beta} \right]^{\beta} > 0$$

then from Eq. (52) and condition (53), we have $\dot{f}_{rw} \geq 0$ if $f_{rw} \leq L$, $r \in R_w$, $w \in W$. Therefore, the set (49) is a positively invariant set of the path flows. This completes the proof.

□

Now we are ready to prove Theorem 6.

**Proof of Theorem 6.** Define

$$h_{rw} = c_{rw} f_{rw}^{1/\beta} \tag{54}$$

then

$$\ln h_{rw} = \ln c_{rw} + \frac{1}{\beta} \ln f_{rw}$$

and Eq. (47) can be rewritten as

$$\dot{f}_{rw} = \frac{\beta\eta}{d_w} \frac{1}{p_{kw} f_{kw}^{1/\beta}} \sum_{s \in R_w} f_{rw} f_{sw} \left( h_{sw} - h_{rw} \right) \tag{55}$$

Further define

$$V_3\left(f\right)=\sum_{a\in A}\int_0^{v_a(f)}\ln c_a\left(\omega\right)\mathrm{d}\omega+\frac{1}{\beta}\sum_{w\in W}\sum_{r\in R_w}f_{rw}\left(\ln f_{rw}-1\right) \tag{56}$$

which is the objective function of the mathematical programming formulation used by Kitthamkesorn and Chen (2013) for solving weibit SUE with separable and multiplicative link travel cost functions. Then we have

$$\frac{\partial V_3\left(f\right)}{\partial f_{rw}}=\ln c_{rw}+\frac{1}{\beta}\ln f_{rw}=\ln h_{rw} \tag{57}$$

and

$$\begin{aligned}\dot{V}_3\left(f\right)&=\sum_{w\in W}\sum_{r\in R_w}\frac{\partial V_3\left(f\right)}{\partial f_{rw}}\dot{f}_{rw}\\&=\sum_{w\in W}\sum_{r\in R_w}\ln h_{rw}\frac{\beta\eta}{d_w}\frac{1}{p_{kw}f_{kw}^{1/\beta}}\sum_{s\in R_w}f_{rw}f_{sw}\left(h_{sw}-h_{rw}\right)\\&=\sum_{w\in W}\frac{\beta\eta}{d_w}\frac{1}{p_{kw}f_{kw}^{1/\beta}}\sum_{r\in R_w}\ln h_{rw}\sum_{s\in R_w}f_{rw}f_{sw}\left(h_{sw}-h_{rw}\right)\\&=-\frac{1}{2}\sum_{w\in W}\frac{\beta\eta}{d_w}\frac{1}{p_{kw}f_{kw}^{1/\beta}}\sum_{r\in R_w}\sum_{s\in R_w}f_{rw}f_{sw}\left(\ln h_{rw}-\ln h_{sw}\right)\left(h_{rw}-h_{sw}\right)\end{aligned} \tag{58}$$

where the second equality is obtained by substituting Eqs. (55) and (57), and the last equality is obtained by swapping the summation indices $r$ and $s$ on half of its left-hand side. In Eq. (58), as $f>0$, $p\triangleq\left(p_{rw},r\in R_w,w\in W\right)^{\mathrm{T}}>0$, and the $\ln\left(\cdot\right)$ function is strictly increasing, then we have $\dot{V}_3\left(f\right)\leq 0$, with the equality holding iff

$$h_{rw}=h_{sw},\ \ \forall\ r,s\in R_w,\ \ w\in W \tag{59}$$

which, referring to Eqs. (54) and (39), indicates weibit SUE. Further, according to Eq. (55), any path flow pattern satisfying condition (59) will remain unchanged. Hence the largest invariant set in $\Phi\triangleq\left\{f\middle|\dot{V}_3\left(f\right)=0,\ f\in\Omega\right\}$ is itself, i.e., the set of weibit-SUE states.

Further, according to Lemma 2, given any $L'\in\left(0,L\right]$, the compact set (49) is a positively invariant set of the path flows. Then by definition (50) and Eq. (1), given any $\varepsilon \in \left(0, Q_{\inf}\right]$, the compact set

$$P \triangleq \left\{ p \middle| p \geq \varepsilon \right\} \tag{60}$$

is a positively invariant set of $p$. Since the right-hand side of dynamic (1) & (47) is differentiable and thus Lipschitz continuous with respect to $f \in \Omega$ and $p > 0$, then by LaSalle's theorem, if $f \in \tilde{\Omega}(L')$, $p \in P$ and $\tilde{\Omega}(L') \cap \Phi \neq \varnothing$, then $f(t) \to \tilde{\Omega}(L') \cap \Phi$ as $t \to +\infty$. As in the definitions (49) and (60) of the two sets, $L'$ and $\varepsilon$ can be arbitrarily close to zero, this actually means that, if $f \in \Omega$ and $p > 0$, then $f(t) \to \lim_{L' \to 0}\left(\tilde{\Omega}(L') \cap \Phi\right) = \Phi$ as $t \to +\infty$, i.e., the path flow pattern will converge to weibit SUE. Consequently, according to Eq. (1), the perceived costs will converge to the weibit-SUE path costs. This completes the proof. □

## 5. Numerical examples

So far, we have proved the global stability of several day-to-day models, by considering different assumptions for different models. In this section, we will use numerical examples to further investigate the global stability of these models under relaxed or more common assumptions (more specifically, under nonseparable and monotone cost functions), which was not rigorously proved in the previous sections. As the global stability of the logit-FIFO and logit-ESL dynamics have been proved for the general nonseparable and monotone cost functions, they will not be investigated in this part.

The numerical studies are conducted on a simple network illustrated by Figure 4. There is only one OD pair with a fixed demand of 10 served by three paths (Path 1, O→1→2→D; Path 2, O→3→4→D; Path 3, O→3→5→2→D). All numerical examples will consider nonseparable link cost functions of the form $c_i = c_i^0\left[1+0.15\left(v_i/Y_i\right)^4\right]+\sum_{j=1}^{5} m_{ij}v_j$, where $c_i^0$ and $Y_i$ are parameters given in Table 1, and

$$\left[m_{ij}\right]_{5\times5}=\begin{bmatrix}1 & 1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

It is not difficult to verify that the specified nonseparable link cost functions are (strictly) monotone. If the link costs are also additive, then the path cost functions will be monotone. Therefore, in the following investigations, we assume the link costs are additive. The parameters of the day-to-day models are chosen as $\eta=1$, $\beta=0.5$ and $\alpha=1$. The continuous-time ordinary differential equation sets representing the day-to-day dynamics are solved numerically by the fourth-order Runge-Kutta method using Matlab's built-in function *ode45*.

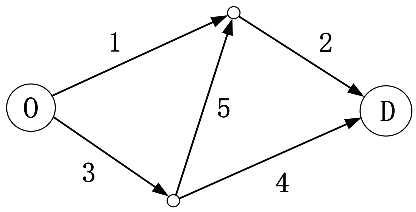


**Figure 4.** The network structure.

**Table 1.** Parameters of the link travel cost functions

| Link index $i$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $c_i^0$ | 2 | 1 | 1 | 2 | 1 |
| $Y_i$ | 3 | 7 | 7 | 3 | 4 |

First, we examine the weibit-FIFO dynamic and type-I weibit-ESL dynamic. In Theorem 5, these two dynamics have been proved to be globally stable under log-monotone path cost functions without requiring the link costs to be separable or additive/multiplicative. Here we further investigate their stability under monotone path cost functions based on the additive and nonseparable link cost functions specified earlier. The weibit-FIFO form (43) is used to calculate the flow trajectories, which are illustrated in Figure 5. The trajectories starting from different initial states all converge to the same path flow pattern, indicating the model is globally stable under the specific network structure and the specific monotone path cost functions.

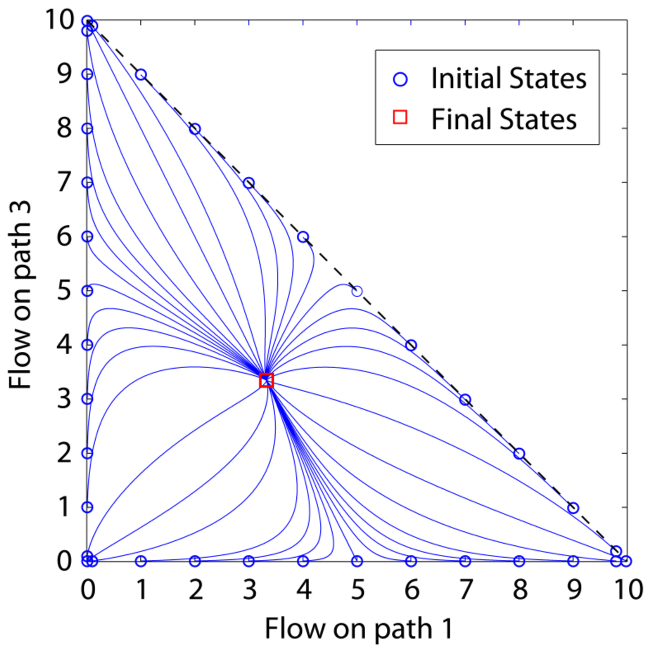


**Figure 5.** Flow evolution of weibit-FIFO dynamic (43)

Theorem 6 has shown that the type-II weibit-ESL dynamic is globally stable under separable and multiplicative link travel cost functions. In this numerical example, the case of nonseparable and additive link costs is studied. The form (1) & (44) is used to generate the trajectories. The trajectories of path flows are displayed in Figure 6. Since the state variables in form (1) & (44) are the perceived costs, then to ensure that the evolution starts from the specific flow patterns shown in Figure 6, given any particular initial path flow pattern $f^0 = \left( f_1^0, f_2^0, f_3^0 \right)^{\mathrm{T}}$, the initial perceived cost $p_i^0$ for each path $i \in \{1,2,3\}$ is calculated as $p_i^0 = \left( \left( \max_{k \in \{1,2,3\}} f_k^0 \right) \big/ f_i^0 \right)^{\frac{1}{\beta}}$. The trajectories displayed in Figure 6 imply that the system is globally stable.

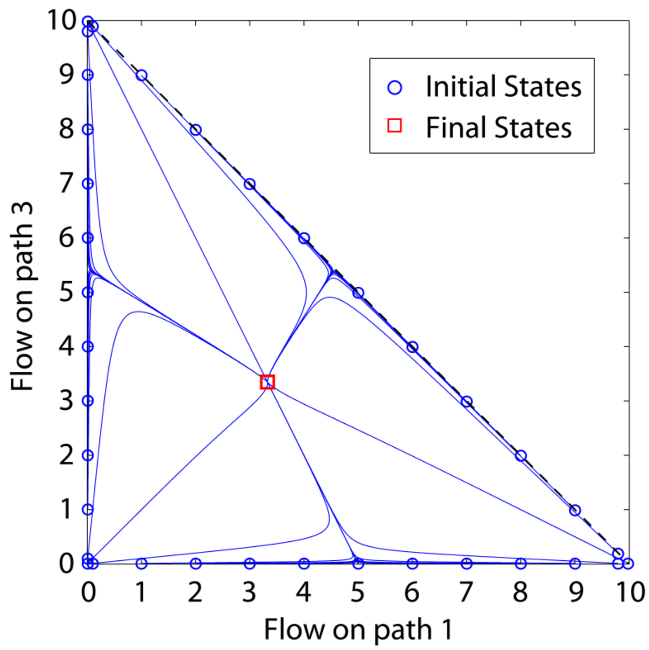


**Figure 6.** Flow evolution of type-II weibit-ESL dynamic (1) & (44)

Finally, we examine Cantarella and Cascetta (1995)'s model (1) & (23). From Theorem 3, we know that the model is globally stable under separable and additive link cost functions. Here we further look into the nonseparable case. The trajectories of flows and perceived costs are generated based on formulation (1) & (23) and displayed in Figure 7 and Figure 8, where the initial perceived costs are set to be six times the value of the initial flows on the same paths. As Figure 7 shows, the flows fluctuate a lot but all converge to the same pattern. In Figure 8, the trajectories of perceived costs on paths 1 and 3 also converge to the same point; in this case, the perceived cost on path 2, although not included in the figure, will also converge, according to Eq. (23).

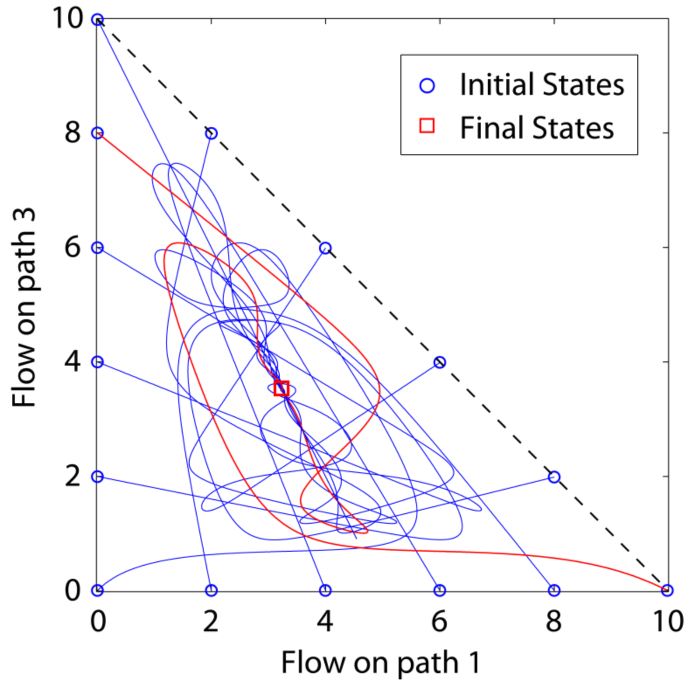


**Figure 7.** Flow trajectories of Cantarella and Cascetta (1995)'s model (1) & (23) (both the blue and red lines indicate trajectories; the red color is used to highlight the trajectories which start from two arbitrarily chosen initial states)

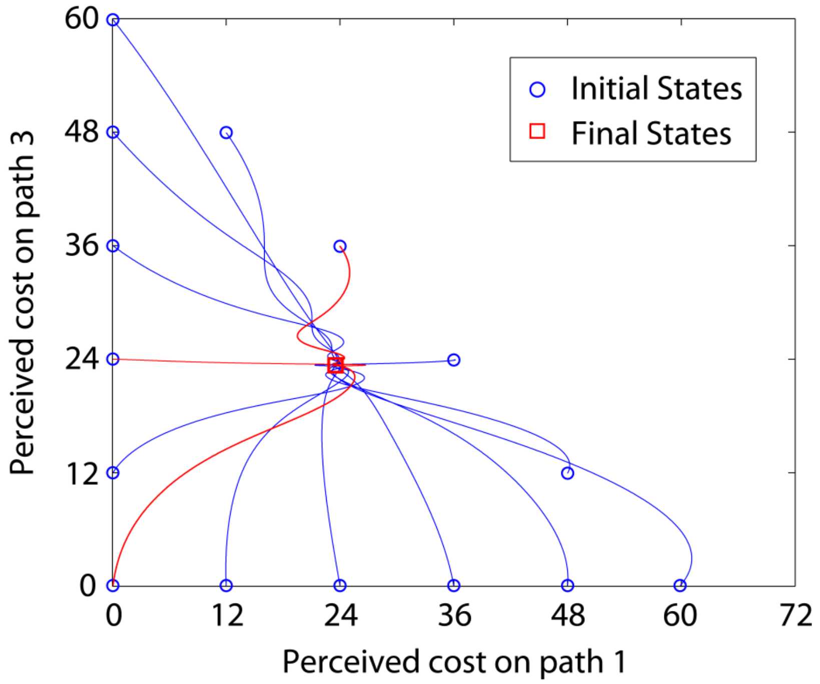


**Figure 8.** Trajectories of perceived path costs of Cantarella and Cascetta (1995)'s model (1) & (23) (both the blue and red lines indicate trajectories; the red color is used to highlight the trajectories which start from three arbitrarily chosen initial states)

## 6. Conclusions and future research

In this paper, we studied the interconnection of some day-to-day models with and without the perceived-cost terms in their formulation. The main results and the relationship between these models are summarized in Figure 9, where each block represents a day-to-day model. The dashed-line blocks represent existing models in the literature, and the solid-line blocks represent the new models proposed in this paper. Rectangles with sharp corners represent models without perceived costs in the model formulations, and round corners the models with perceived costs. The bidirectional arrow means the two models are equivalent, while the unidirectional arrow indicates that the model at the head of the arrow is an extension of the model at the tail. The grey blocks mean the dynamics are proved to be globally stable under separable link cost functions, and white blocks when the cost functions can be nonseparable (except for the FIFO dynamic and replicator dynamic as their stability was not investigated in this paper).

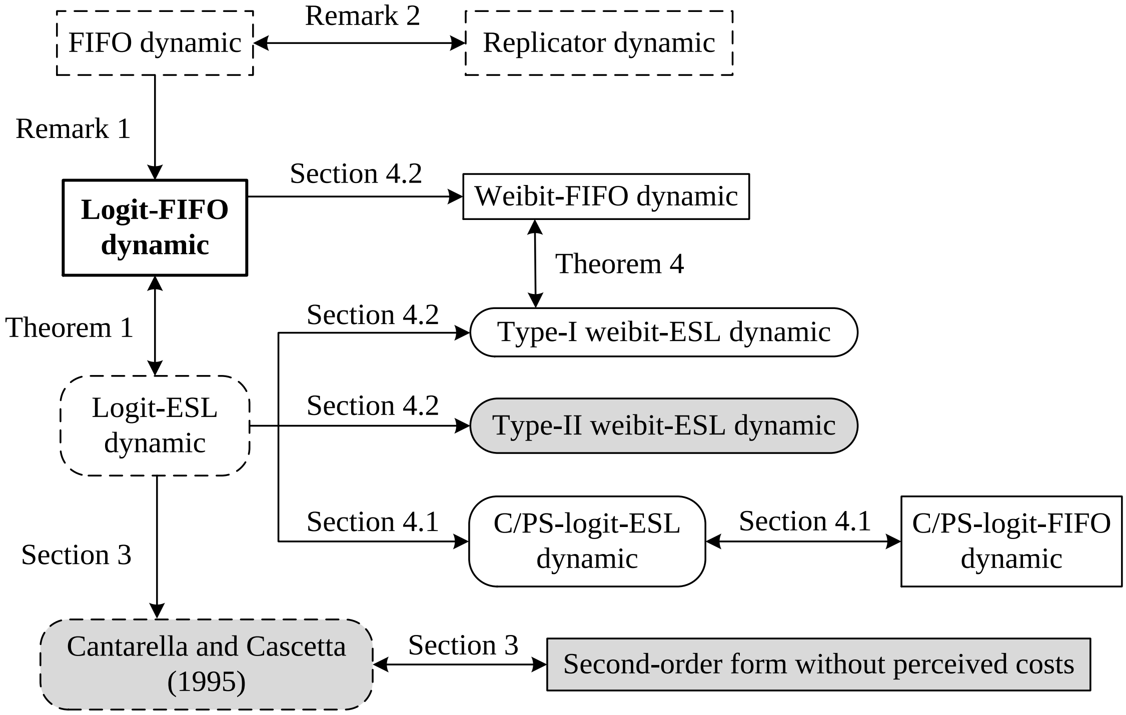


**Figure 9.** Main results and model relationship in this paper

Specifically, we showed that the logit-ESL dynamic (Watling, 1999) is equivalent to a logit-based extension of the FIFO dynamic (Jin, 2007). Via such equivalence, we proved the global stability of the logit-ESL dynamic under nonseparable and monotone travel cost functions. We also showed that Cantarella and Cascetta (1995)'s model, which can be treated as a combination of the logit-ESL dynamic and the logit dynamic, is equivalent to a second-order dynamic of flows, and proved its global stability under separable travel cost functions. Further, we introduced other discrete choice modeling concepts into the logit-ESL model and formulated new day-to-day models with different fixed points including C/PS-logit SUE and weibit SUE. These new models were also reformulated into other forms and proved to be globally stable under various conditions:

- The C/PS-logit-ESL and C/PS-logit-FIFO dynamics are globally stable under nonseparable and monotone cost functions;
- The type-I weibit-ESL dynamic is equivalent to the weibit-FIFO dynamic and globally stable when the path cost functions are log-monotone;
- The type-II weibit-ESL dynamic is globally stable when the link costs are separable and multiplicative.

Finally, we used numerical examples to study the global stability of several day-to-day models under more general/common assumptions, which was not rigorously proved in the theoretical sections. It was observed that the weibit-FIFO dynamic (or equivalently the type-I webit-ESL dynamic), the type-II weibit-ESL dynamic and Cantarella and Cascetta (1995)'s model are all globally stable when the travel cost functions are nonseparable, monotone and additive.

This paper enriches our understanding on the day-to-day dynamics, but also raises many open questions. First, in the formulations of logit-FIFO dynamic as well as other logit-based day-to-day models such as those in Smith and Watling (2016) and Xiao et al. (2019), the logarithms of path flows are included as additional costs to the actual path costs; there is still no convincing explanation on what the logarithmic path flow represents from a behavioral point of view, or how to derive it from the Gumbel-distribution assumption. Second, when proving the global stability of the weibit-based models, we required the link costs to be multiplicative or the path cost functions to be log-monotone. This is different from the traditional assumptions on link travel costs which are additive, or on path cost functions which are monotone; however, the numerical examples seem to imply that these traditional assumptions can still guarantee the global stability of the weibit-based models. Therefore, the second question for the future is to prove the global stability of the weibit-based dynamics under additive and monotone cost functions. Third, to prove the global stability of the type-II weibit-ESL dynamic and Cantarella and Cascetta (1995)'s model, we required separable travel cost functions; hopefully this restriction can be lifted in the future to allow nonseparable cost functions. Last but not least, since the cognitive mechanism of the logit/weibit-FIFO dynamic can be explained by their equivalent which includes perceived costs, it would be interesting if we can find the equivalent forms of other day-to-day models such as the logit-based PSAP (Smith and Watling, 2016).

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## Appendix A. Definitions of stability

**Definition A1**. *Consider an autonomous system*

$$\dot{x} = z(x) \tag{61}$$

*where* $z : M \to \mathbb{R}^m$ *is a locally Lipschitz map from a domain* $M \subset \mathbb{R}^m$ *into* $\mathbb{R}^m$. *Suppose* $x^*$ *is an equilibrium point of (61), then* $x^*$ *is*

- *locally stable if* $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$, *such that*

$$\left\| x(0) - x^* \right\| < \delta \;\Rightarrow\; \left\| x(t) - x^* \right\| < \varepsilon \qquad \forall t \geq 0$$

- *locally asymptotically stable if it is locally stable and* $\delta$ *can be chosen such that*

$$\left\| x(0) - x^* \right\| < \delta \;\Rightarrow \lim_{t \to \infty} \left\| x(t) - x^* \right\| = 0$$

- *globally asymptotically stable if*

$$x(0) \in M \;\Rightarrow \lim_{t \to \infty} \left\| x(t) - x^* \right\| = 0$$

## Appendix B. LaSalle's theorem

**Definition B1**. (pp. 127, Section 4.2, Khalil, 2002) *A set* $\Omega$ *is said to be an invariant set with respect to (61) if*

$$x(0) \in \Omega \Rightarrow x(t) \in \Omega, \;\; \forall t \in \mathbb{R}$$

$\Omega$ *is said to be a positively invariant set if*

$$x(0) \in \Omega \Rightarrow x(t) \in \Omega, \;\; \forall t \geq 0$$

The global stability of system (61) can be investigated by LaSalle's theorem below.

**Theorem B1.** (Theorem 4.4, Khalil, 2002) *Let* $\Omega \subset M$ *be a compact (closed and bounded) set that is positively invariant with respect to the dynamical system (61), and* $V : M \to \mathbb{R}$ *be a continuously differentiable function such that* $\dot{V}(x) \leq 0$ *in* $\Omega$. *Let* $\Pi$ *be the set of all points in* $\Omega$ *where* $\dot{V}(x) = 0$. *Let G be the largest invariant set in* $\Pi$. *Then every solution of (61) starting in* $\Omega$ *approaches G as* $t \to \infty$.

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