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New Formulations and Solution Approaches for Train ***Eco-driving*** Problems

[https://ssrn.com/abstract=4928445]

(Join work with Zhuang Xiao & Edward Chung)

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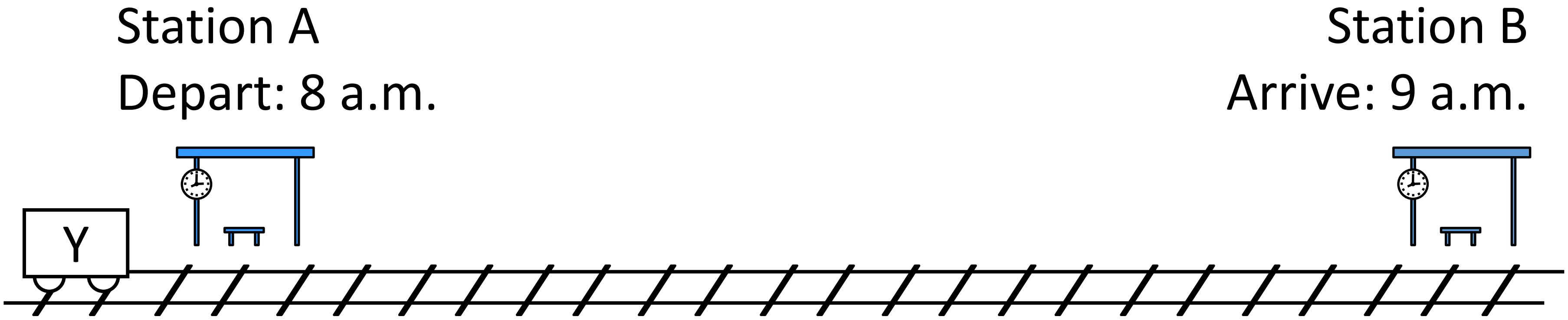
Outline

- What is eco-driving
- Formulation of eco-driving problem
- Existing solution methods
- Our solution method
    - Reformulation, relaxation, and valid inequalities
- Possible directions for future research

What is Eco-driving?



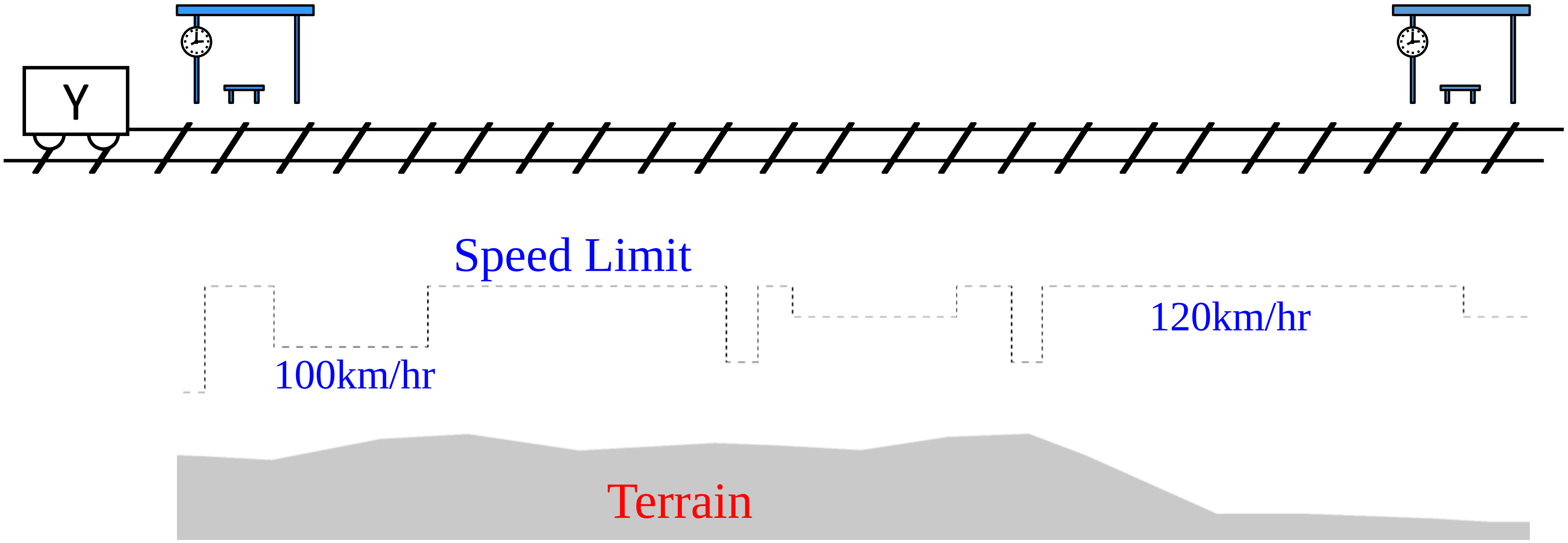
How to drive the train (i.e., control the power and brake) so as to

- Arrive at the station on time
- Consume minimum amount of energy

Factors Considered in Eco-driving: **Track Condition**

Station A
Depart: 8 a.m.

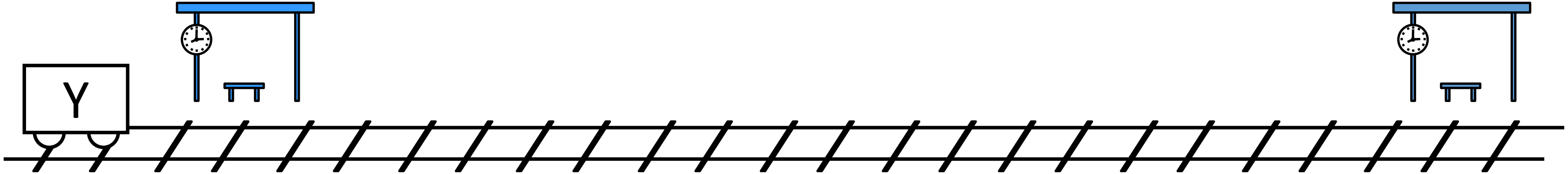
Station B
Arrive: 9 a.m.

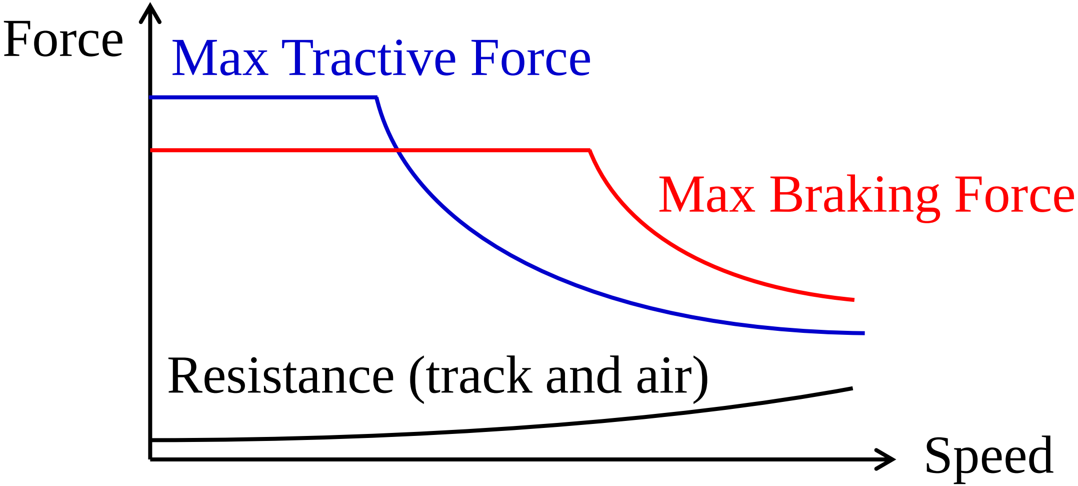


Factors Considered in Eco-driving: Train Characteristics

Station A
Depart: 8 a.m.

Station B
Arrive: 9 a.m.

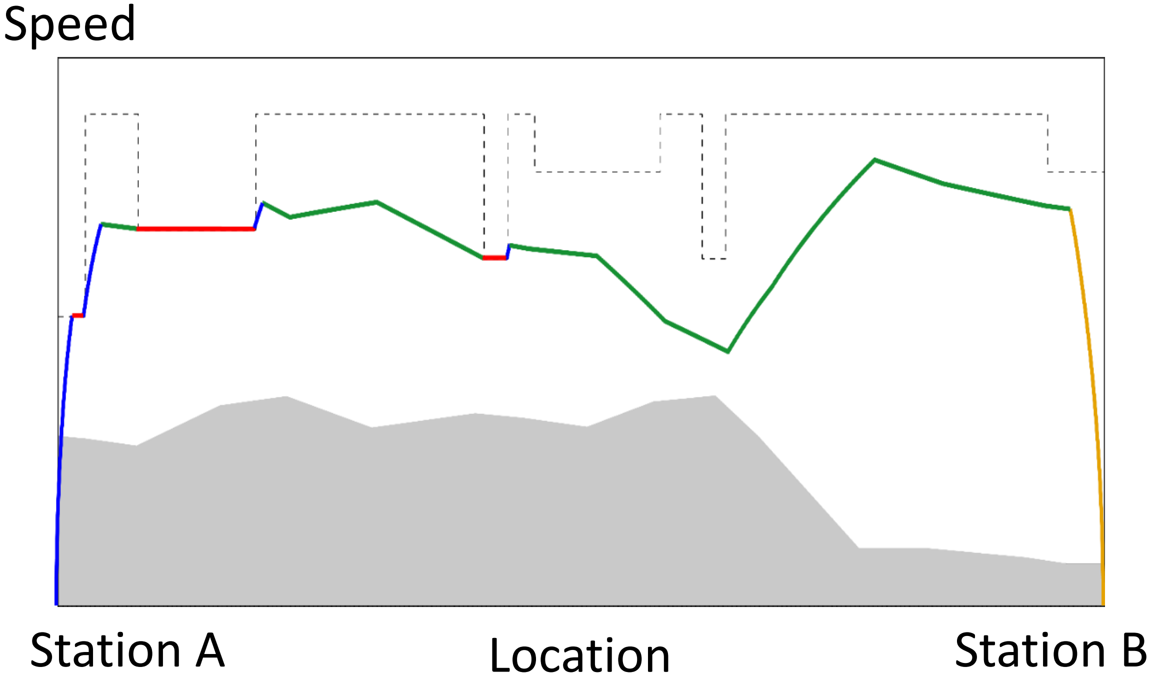




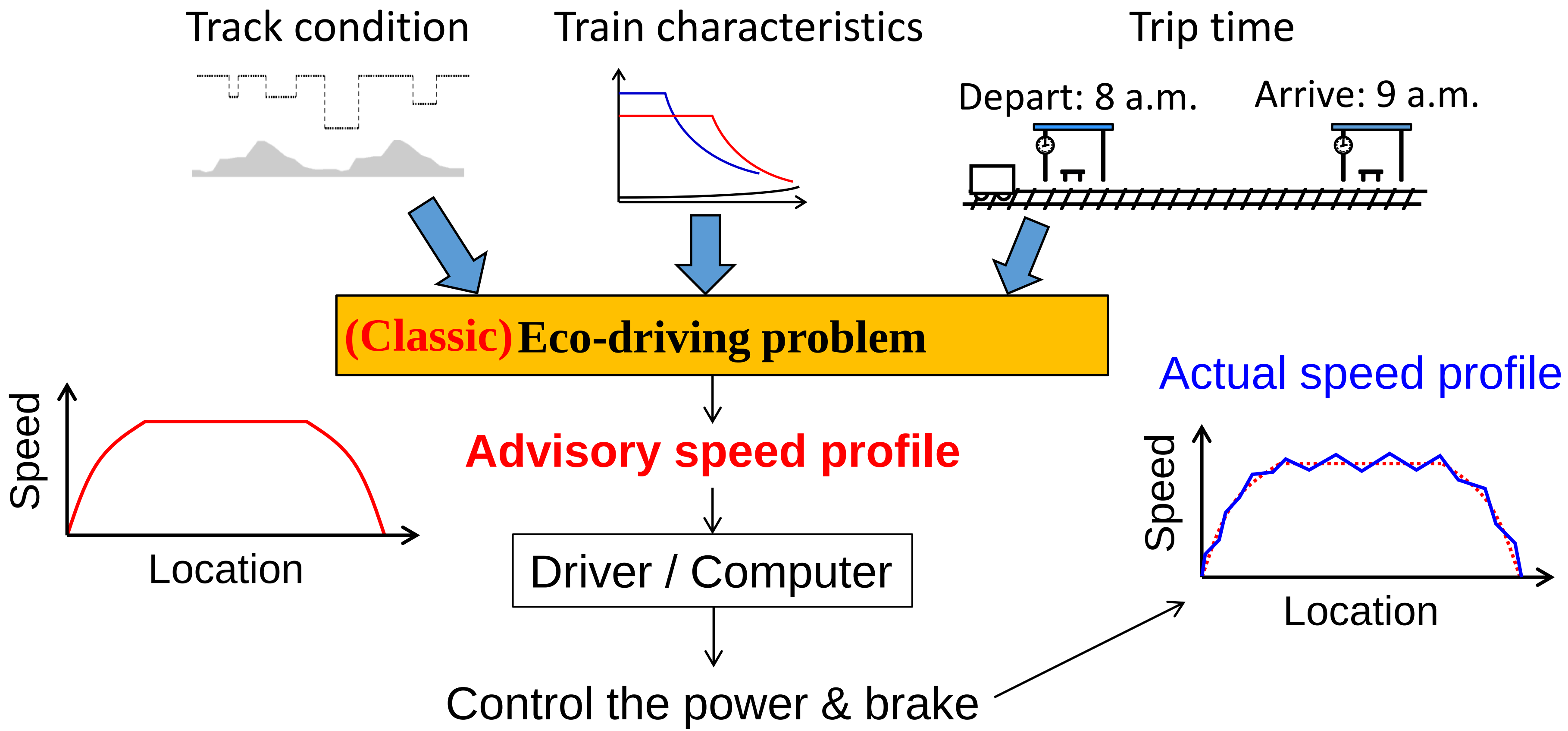
Result of Solving an Eco-driving Problem

Solving the eco-driving problem, we can get a speed profile that

- Guarantees the train reaching next station on time,
- Minimizes the energy consumption



Speed Tracking



Track condition

Train characteristics

Trip time

Depart: 8 a.m.

Arrive: 9 a.m.

(Classic) Eco-driving problem

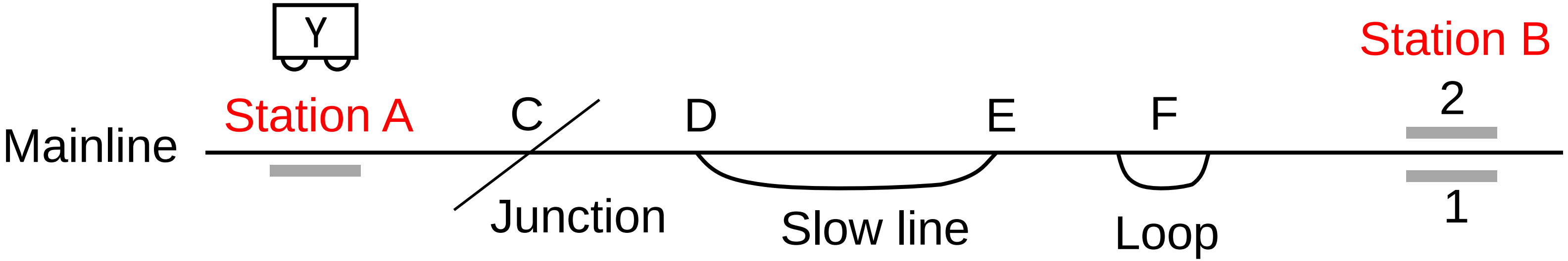
Advisory speed profile

Driver / Computer

Control the power & brake

Actual speed profile

Other Factors Considered in Eco-driving: **Schedule**



- What passengers see

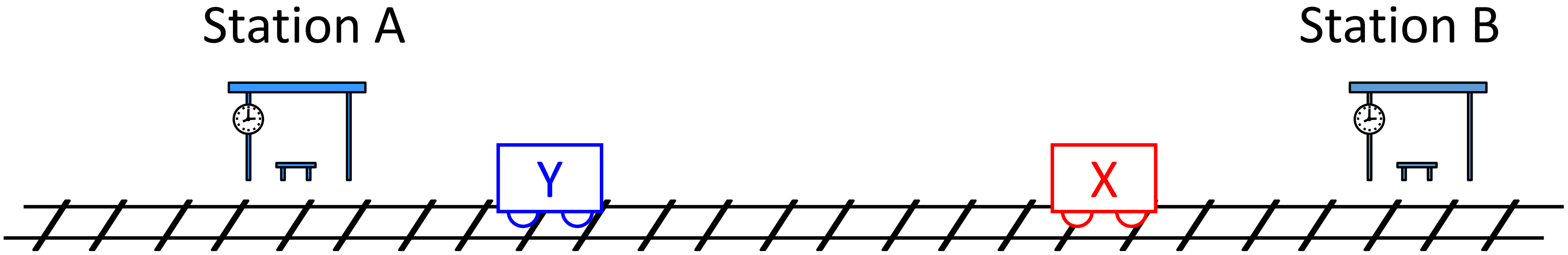
Train Y	Arr	Dep
Station A	-	8:00
Station B (platform 2)	9:00	-

- What train operators/drivers see

Train Y	Arr	Dep
A	-	8:00
C	[8:13, 8:17]	
D (slow)	8:25	
E	8:35	
F (mainline)	8:40	
B (platform 2)	9:00	-

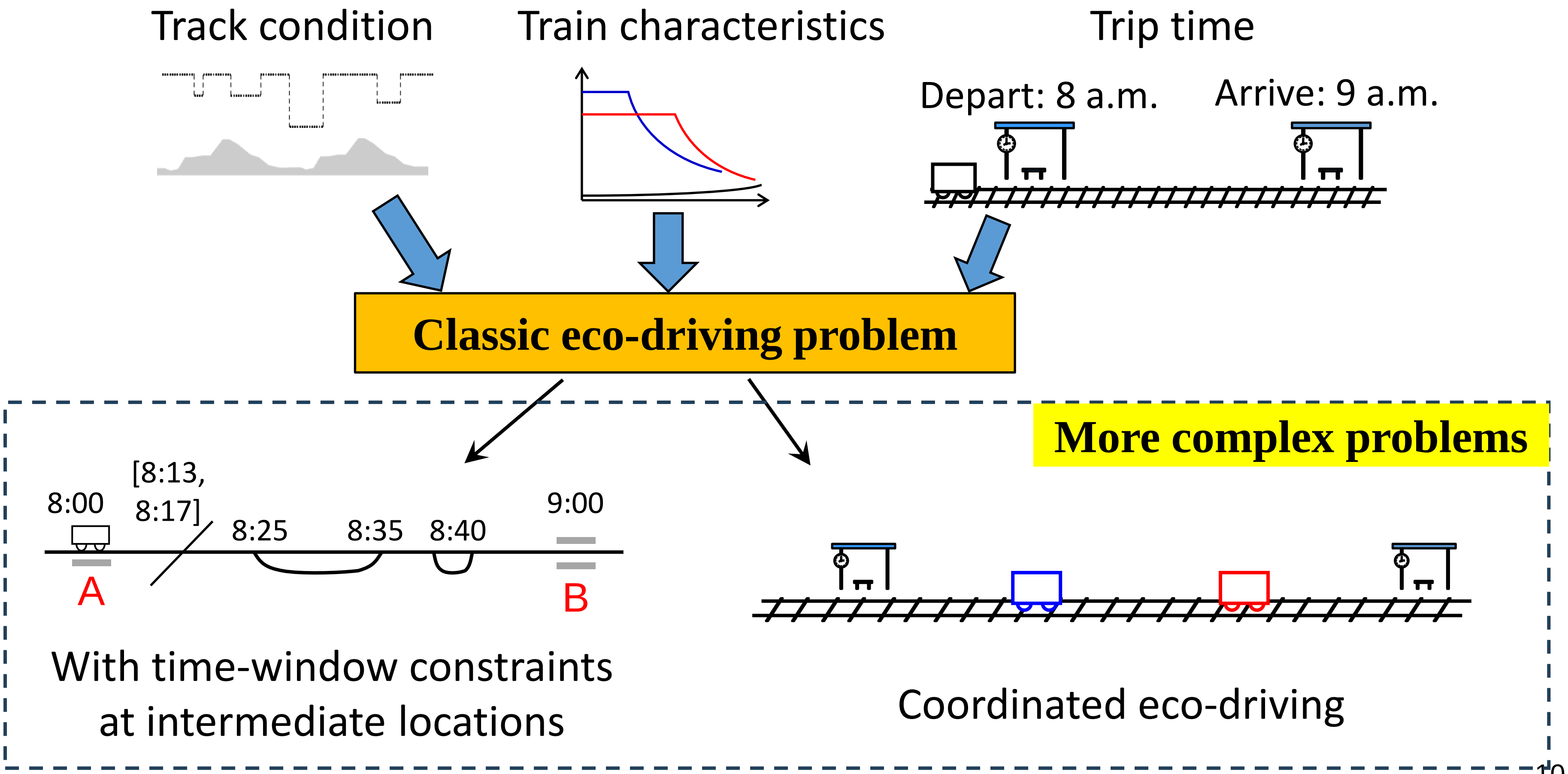
non-stop

Other Factors Considered in Eco-driving: Train Interference



- Train Y must not **collide** with train X
- The eco-driving of trains X and Y can be **coordinated**

Type of Eco-driving Problems



Track condition

Train characteristics

Trip time

Depart: 8 a.m.

Arrive: 9 a.m.

Classic eco-driving problem

More complex problems

With time-window constraints at intermediate locations

Coordinated eco-driving

Formulation of Classic Eco-driving Problem

$$\min_{F(s):s\in[S_0,S_f]} \int_{S_0}^{S_f} \max\{F(s), \eta_{\mathrm{reg}} F(s)\} \, \mathrm{d}s$$

s.t.

$$\frac{\mathrm{d}v(s)}{\mathrm{d}s} = \frac{F(s) - [av(s)^2 + bv(s) + c] - mg \sin\alpha(s)}{m \cdot v(s)}$$

$$\frac{\mathrm{d}t(s)}{\mathrm{d}s} = \frac{1}{v(s)}$$

$$\varepsilon \leq v(s) \leq \bar{V}(s)$$

$$F_{\min} \leq F(s) \leq F_{\max}$$

$$P_{\min} \leq F(s)v(s) \leq P_{\max}$$

$$t(S_f) - t(S_0) \leq T$$

$$v(S_0) = V_0, v(S_f) = V_f$$

Optimal control problem

Formulation of Classic Eco-driving Problem

$$\min_{F(s):s\in[S_0,S_f]} \int_{S_0}^{S_f} \max\{F(s), \eta_{\text{reg}} F(s)\}\,\mathrm{d}s$$

Total energy consumption of the trip

s.t.

$$\frac{\mathrm{d}v(s)}{\mathrm{d}s} = \frac{F(s) - [av(s)^2 + bv(s) + c] - mg\sin\alpha(s)}{m \cdot v(s)}$$

$$\frac{\mathrm{d}t(s)}{\mathrm{d}s} = \frac{1}{v(s)}$$

$$\varepsilon \leq v(s) \leq \bar{V}(s)$$

$$F_{\min} \leq F(s) \leq F_{\max}$$

$$P_{\min} \leq F(s)v(s) \leq P_{\max}$$

$$t(S_f) - t(S_0) \leq T$$

$$v(S_0) = V_0, v(S_f) = V_f$$

s: Location

$F(s)$: Applied force at location s

$\eta_{\text{reg}} \in [0,1)$: Proportion of braking energy reused

S_0 & S_f: Trip start & end position

Formulation of Classic Eco-driving Problem

$$\min_{F(s):s\in[S_0,S_f]} \int_{S_0}^{S_f} \max\{F(s), \eta_{\mathrm{reg}}F(s)\}\,\mathrm{d}s$$

s.t.

$$\frac{\mathrm{d}v(s)}{\mathrm{d}s} = \frac{F(s) - [av(s)^2 + bv(s) + c] - mg\sin\alpha(s)}{m \cdot v(s)}$$

$$\frac{\mathrm{d}t(s)}{\mathrm{d}s} = \frac{1}{v(s)}$$

$$\varepsilon \le v(s) \le \bar{V}(s)$$

$$F_{\min} \le F(s) \le F_{\max}$$

$$P_{\min} \le F(s)v(s) \le P_{\max}$$

$$t(S_f) - t(S_0) \le T$$

$$v(S_0) = V_0, v(S_f) = V_f$$

$t(s)$: clock time at location s

$v(s)$: speed at location s

Formulation of Classic Eco-driving Problem

Resistance due to track & air

Component of gravitational force

$$\min_{F(s):s\in[S_0,S_f]} \int_{S_0}^{S_f} \max\{F(s),\eta_{\text{reg}}F(s)\}\,\mathrm{d}s$$

$$\text{s.t.}\quad \frac{\mathrm{d}v(s)}{\mathrm{d}s}=\frac{\mathrm{d}v(s)}{\mathrm{d}t(s)}\frac{\mathrm{d}t(s)}{\mathrm{d}s}=\frac{F(s)-\underline{[av(s)^2+bv(s)+c]}-\underline{mg\sin\alpha(s)}}{m}\frac{1}{v(s)}$$

$$\frac{\mathrm{d}t(s)}{\mathrm{d}s}=\frac{1}{v(s)}$$

Train mass

$\alpha(s)$: gradient at location s

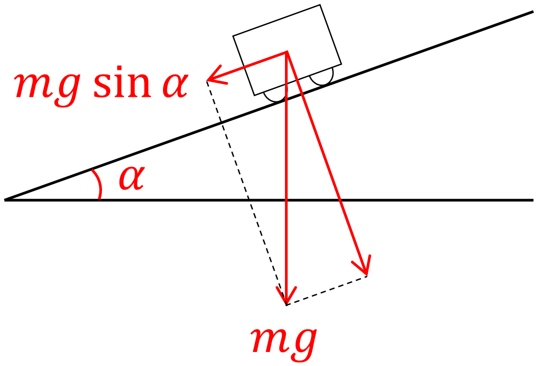
$$\varepsilon\le v(s)\le \bar{V}(s)$$

$$F_{\min}\le F(s)\le F_{\max}$$

$$P_{\min}\le F(s)v(s)\le P_{\max}$$

$$t(S_f)-t(S_0)\le T$$

$$v(S_0)=V_0, v(S_f)=V_f$$



Formulation of Classic Eco-driving Problem

$$\min_{F(s):s\in[S_0,S_f]} \int_{S_0}^{S_f} \max\{F(s), \eta_{\text{reg}} F(s)\}\,\mathrm{d}s$$

s.t.

$$\frac{\mathrm{d}v(s)}{\mathrm{d}s} = \frac{F(s) - [av(s)^2 + bv(s) + c] - mg\sin\alpha(s)}{m \cdot v(s)}$$

$$\frac{\mathrm{d}t(s)}{\mathrm{d}s} = \frac{1}{v(s)}$$

$\varepsilon \leq v(s) \leq \bar{V}(s)$ Speed limit constraint

$$F_{\min} \leq F(s) \leq F_{\max}$$

$$P_{\min} \leq F(s)v(s) \leq P_{\max}$$

$$t(S_f) - t(S_0) \leq T$$

$$v(S_0) = V_0, v(S_f) = V_f$$

$v(s)$: speed at location s

$\bar{V}(s)$: speed limit at location s

ε: small positive number

Formulation of Classic Eco-driving Problem

$$\min_{F(s):s\in[S_0,S_f]} \int_{S_0}^{S_f} \max\{F(s), \eta_{\mathrm{reg}} F(s)\}\, \mathrm{d}s$$

s.t.

$$\frac{\mathrm{d}v(s)}{\mathrm{d}s} = \frac{F(s) - [av(s)^2 + bv(s) + c] - mg\sin\alpha(s)}{m \cdot v(s)}$$

$$\frac{\mathrm{d}t(s)}{\mathrm{d}s} = \frac{1}{v(s)}$$

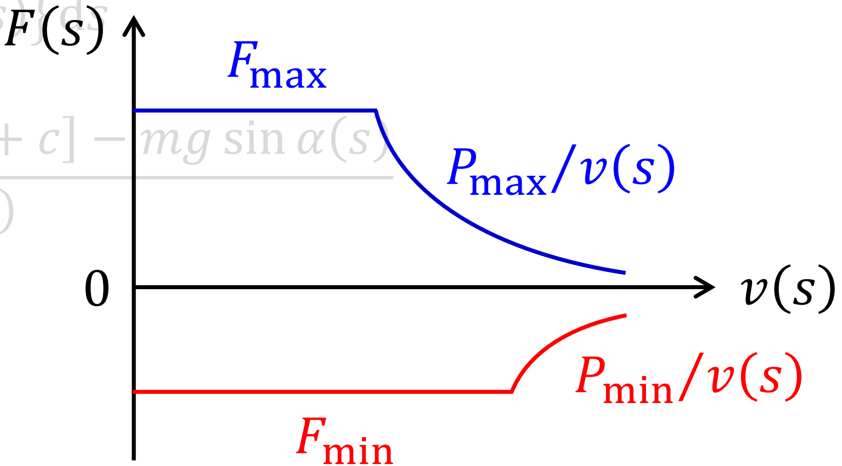
$$\varepsilon \le v(s) \le \bar{V}(s)$$

$$F_{\min} \le F(s) \le F_{\max}$$

$$P_{\min} \le F(s)v(s) \le P_{\max}$$

$$t(S_f) - t(S_0) \le T$$

$$v(S_0) = V_0, v(S_f) = V_f$$



Power and brake capacity constraints

$F_{\max}$, $P_{\max}$: max force/power

$F_{\min}$, $P_{\min}$: min force/power

Formulation of Classic Eco-driving Problem

$$\min_{F(s):s\in[S_0,S_f]} \int_{S_0}^{S_f} \max\{F(s), \eta_{\text{reg}} F(s)\} \, \mathrm{d}s$$

s.t.

$$\frac{\mathrm{d}v(s)}{\mathrm{d}s} = \frac{F(s) - [av(s)^2 + bv(s) + c] - mg\sin\alpha(s)}{m \cdot v(s)}$$

$$\frac{\mathrm{d}t(s)}{\mathrm{d}s} = \frac{1}{v(s)}$$

$$\varepsilon \le v(s) \le \bar{V}(s)$$

$$F_{\min} \le F(s) \le F_{\max}$$

$$P_{\min} \le F(s)v(s) \le P_{\max}$$

$$t(S_f) - t(S_0) \le T$$

Total trip time constraint

$$v(S_0) = V_0, v(S_f) = V_f$$

$t(S_f)$, $t(S_0)$: Clock time at trip end & start

T: Scheduled travel time of the trip

Formulation of Classic Eco-driving Problem

$$\min_{F(s):s\in[S_0,S_f]} \int_{S_0}^{S_f} \max\{F(s), \eta_{\text{reg}} F(s)\}\,\mathrm{d}s$$

s.t.

$$\frac{\mathrm{d}v(s)}{\mathrm{d}s} = \frac{F(s) - [av(s)^2 + bv(s) + c] - mg\sin\alpha(s)}{m \cdot v(s)}$$

$$\frac{\mathrm{d}t(s)}{\mathrm{d}s} = \frac{1}{v(s)}$$

$$\varepsilon \le v(s) \le \bar{V}(s)$$

$$F_{\min} \le F(s) \le F_{\max}$$

$$P_{\min} \le F(s)v(s) \le P_{\max}$$

$$t(S_f) - t(S_0) \le T$$

$$v(S_0) = V_0, v(S_f) = V_f$$

Required speed at trip start and end

Solution Methods of Eco-driving Problems

$$\min_{F(s):s\in[S_0,S_f]} \int_{S_0}^{S_f} \max\{F(s), \eta_{\text{reg}} F(s)\}\, \mathrm{d}s$$

s.t.

$$\frac{\mathrm{d}v(s)}{\mathrm{d}s} = \frac{F(s) - [av(s)^2 + bv(s) + c] - mg\sin\alpha(s)}{m \cdot v(s)}$$

$$\frac{\mathrm{d}t(s)}{\mathrm{d}s} = \frac{1}{v(s)}$$

$$\varepsilon \le v(s) \le \bar{V}(s)$$

$$F_{\min} \le F(s) \le F_{\max}$$

$$P_{\min} \le F(s)v(s) \le P_{\max}$$

$$t(S_f) - t(S_0) \le T$$

$$v(S_0) = V_0, v(S_f) = V_f$$

Optimal control formulation

Two main types of solution approaches

(1) **Indirect** method: Pontryagin's maximum principle (+ numerical method)

– challenge to analyze, especially the complex problems

(2) **Direct** method : via discretization

Direct Method of Solving Classic Eco-driving Problem

$$\min_{F(s):s\in[S_0,S_f]} \int_{S_0}^{S_f} \max\{F(s), \eta_{\text{reg}} F(s)\}\,\mathrm{d}s \Rightarrow \min_{F_k} \sum_{k=0}^{N-1} \max\{F_k, \eta_{\text{reg}} F_k\}\,\Delta_k$$

$$\text{s.t.} \quad \frac{\mathrm{d}v(s)}{\mathrm{d}s} = \frac{F(s) - [av(s)^2 + bv(s) + c] - mg\sin\alpha(s)}{m \cdot v(s)}$$

$$\frac{\mathrm{d}t(s)}{\mathrm{d}s} = \frac{1}{v(s)} \Rightarrow \frac{t_k - t_{k-1}}{\Delta_k} = \frac{1}{v_k}$$

$$\varepsilon \le v(s) \le \bar{V}(s) \Rightarrow \varepsilon \le v_k \le \bar{V}_k$$

$$F_{\min} \le F(s) \le F_{\max}$$

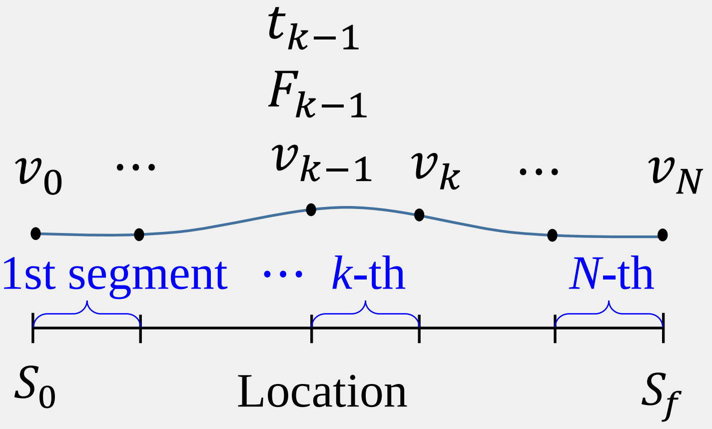
$$P_{\min} \le F(s)v(s) \le P_{\max}$$

$$t(S_f) - t(S_0) \le T$$

$$v(S_0) = V_0, v(S_f) = V_f$$

Discretization

Discretize the whole trip length into N segments. Length of segment k is Δ_k



Direct Method of Solving Classic Eco-driving Problem

$$\min_{F_k} \sum_{k=0}^{N-1} \max\{F_k, \eta_{\text{reg}} F_k\} \Delta_k$$

$$\text{s.t.} \quad \frac{v_k - v_{k-1}}{\Delta_k} = \frac{F_k - (a v_k^2 + b v_k + c) - mg \sin \alpha_k}{m \cdot v_k}$$

$$\frac{t_k - t_{k-1}}{\Delta_k} = \frac{1}{v_k}$$

$$\varepsilon \leq v_k \leq \bar{V}_k$$

$$F_{\min} \leq F_k \leq F_{\max}$$

$$P_{\min} \leq F_k v_k \leq P_{\max}$$

$$t_N - t_0 \leq T$$

$$v_0 = V_0, v_N = V_f$$

Nonconvex optimization

How to solve?

i. Use off-the-shelf solvers

- Cannot guarantee to find exact solutions

Direct Method of Solving Classic Eco-driving Problem

$$\min_{F_k} \sum_{k=0}^{N-1} \max\{F_k, \eta_{\text{reg}} F_k\} \Delta_k$$

$$\text{s.t.} \quad \frac{v_k - v_{k-1}}{\Delta_k} = \frac{F_k - (a v_k^2 + b v_k + c) - mg \sin \alpha_k}{m \cdot v_k}$$

$$\frac{t_k - t_{k-1}}{\Delta_k} = \frac{1}{v_k}$$

$$\varepsilon \leq v_k \leq \bar{V}_k$$

$$F_{\min} \leq F_k \leq F_{\max}$$

$$P_{\min} \leq F_k v_k \leq P_{\max}$$

$$t_N - t_0 \leq T$$

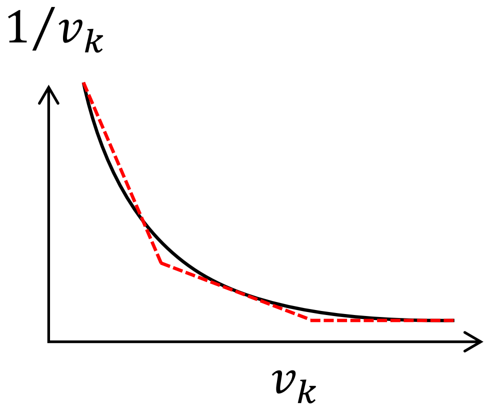
$$v_0 = V_0, v_N = V_f$$

Nonconvex optimization

How to solve?

ii. Approximate the nonlinear terms (⇒ MILP)

- The solution is not the same as that of the nonconvex problem
- More pieces → longer computing time



Direct Method of Solving Classic Eco-driving Problem

$$\min_{F_k} \sum_{k=0}^{N-1} \max\{F_k, \eta_{\text{reg}} F_k\} \Delta_k$$

$$\text{s.t.} \quad \frac{v_k - v_{k-1}}{\Delta_k} = \frac{F_k - (a v_k^2 + b v_k + c) - mg \sin \alpha_k}{m \cdot v_k}$$

$$\frac{t_k - t_{k-1}}{\Delta_k} = \frac{1}{v_k}$$

$$\varepsilon \le v_k \le \bar{V}_k$$

$$F_{\min} \le F_k \le F_{\max}$$

$$P_{\min} \le F_k v_k \le P_{\max}$$

$$t_N - t_0 \le T$$

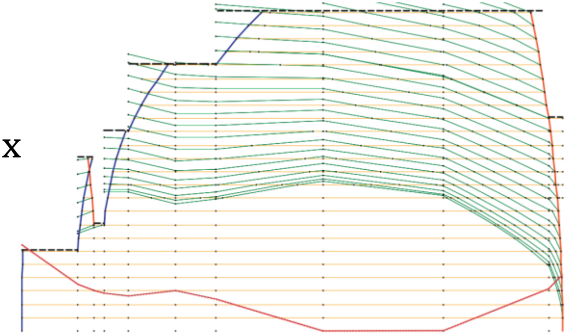
$$v_0 = V_0, v_N = V_f$$

Nonconvex optimization

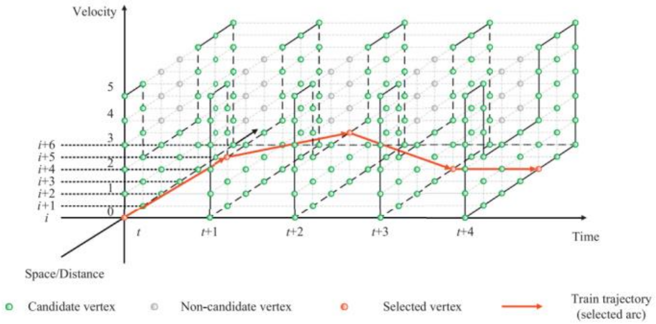
How to solve?

iii. Further discretize speed (and time)

- Similar drawbacks to Method ii



Source: Haahr et al. (2017)



Source: Zhou et al. (2017)

Direct Method of Solving Classic Eco-driving Problem

$$\min_{F_k} \sum_{k=0}^{N-1} \max\{F_k, \eta_{\mathrm{reg}} F_k\} \Delta_k$$

s.t. $$\frac{v_k - v_{k-1}}{\Delta_k} = \frac{F_k - (a v_k^2 + b v_k + c) - mg \sin \alpha_k}{m \cdot v_k}$$ ⇦

Nonconvex optimization

How to solve?

⇨ $$\frac{t_k - t_{k-1}}{\Delta_k} = \frac{1}{v_k}$$

$$\varepsilon \le v_k \le \bar{V}_k$$

$$F_{\min} \le F_k \le F_{\max}$$

⇨ $$P_{\min} \le F_k v_k \le P_{\max}$$

$$t_N - t_0 \le T$$

$$v_0 = V_0, v_N = V_f$$

We aim to:

- Solve this nonconvex problem to exact solutions, and
- Extend our method to solve the more complex eco-driving problems

Our Solution Method: Step 1 – Reformulation

$$\min_{F_k} \sum_{k=0}^{N-1} \max\{F_k, \eta_{\text{reg}} F_k\} \Delta_k$$

$$\text{s.t.} \quad \frac{v_k - v_{k-1}}{\Delta_k} = \frac{F_k - (a v_k^2 + b v_k + c) - mg \sin \alpha_k}{m \cdot v_k}$$

$$\frac{t_k - t_{k-1}}{\Delta_k} = \frac{1}{v_k}$$

$$\varepsilon \le v_k \le \bar{V}_k$$

$$F_{\min} \le F_k \le F_{\max}$$

$$P_{\min} \le F_k v_k \le P_{\max}$$

$$t_N - t_0 \le T$$

$$v_0 = V_0, v_N = V_f$$

Let $z_k = 1/v_k$

$$\frac{t_k - t_{k-1}}{\Delta_k} = z_k$$

Let $E_k = v_k^2/2$

$$\frac{E_k - E_{k-1}}{\Delta_k} = \frac{F_k - (a E_k + b v_k + c) - mg \sin \alpha_k}{m}$$

Our Solution Method: Step 1 – Reformulation

Original: NLP$_{\text{R1}}$

$$\min_{F_k} \sum_{k=0}^{N-1} \max\{F_k, \eta_{\text{reg}} F_k\} \Delta_k$$

s.t. $\frac{E_k - E_{k-1}}{\Delta_k} = \frac{F_k - (aE_k + bv_k + c) - mg \sin \alpha_k}{m}$

$\frac{t_k - t_{k-1}}{\Delta_k} = z_k$

$\varepsilon \le v_k \le \bar{V}_k$

$F_{\min} \le F_k \le F_{\max}$

➡ $P_{\min} \le F_k v_k \le P_{\max}$ → **Bilinear Constraints**

$t_N - t_0 \le T$

$v_0 = V_0, v_N = V_f$

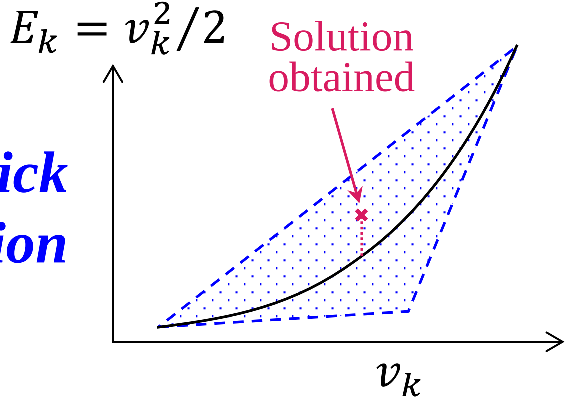
➡ $z_k = 1/v_k$

➡ $E_k = v_k^2/2$

But can be faster !

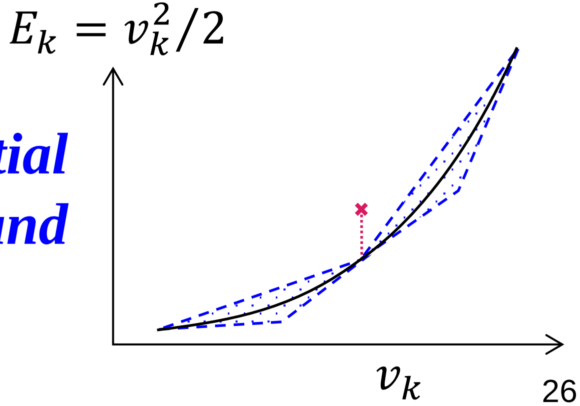
Can be solved to exact solutions by

McCormick relaxation



+

spatial branch-and-bound



Our Solution Method: Step 2 – Relaxation

$$\min_{F_k} \sum_{k=0}^{N-1} \max\{F_k, \eta_{\text{reg}} F_k\} \Delta_k$$

Original: $\mathbf{NLP_{R1}}$

$$\text{s.t.} \quad \frac{E_k - E_{k-1}}{\Delta_k} = \frac{F_k - (aE_k + bv_k + c) - mg \sin \alpha_k}{m}$$

$$\frac{t_k - t_{k-1}}{\Delta_k} = z_k$$

$$\varepsilon \le v_k \le \bar{V}_k$$

$$F_{\min} \le F_k \le F_{\max}$$

⇨ $P_{\min} \le F_k v_k \le P_{\max}$

$$t_N - t_0 \le T$$

$$v_0 = V_0, v_N = V_f$$

⇨ $z_k = 1/v_k \xrightarrow{\text{relaxed}} z_k \ge 1/v_k$

Named $\mathbf{NLP_{R2}}$

⇨ $E_k = v_k^2/2$

The optimal solution of the **relaxed problem** is proved to **satisfy** $\boldsymbol{z_k = 1/v_k}$

Our Solution Method: Step 3 – Valid Inequalities

$$\min_{F_k} \sum_{k=0}^{N-1} \max\{F_k, \eta_{\text{reg}} F_k\} \Delta_k$$

Original: $\mathrm{NLP}_{\mathrm{R1}}$

s.t. $$\frac{E_k - E_{k-1}}{\Delta_k} = \frac{F_k - (aE_k + bv_k + c) - mg \sin \alpha_k}{m}$$

$$\frac{t_k - t_{k-1}}{\Delta_k} = z_k$$

$$\varepsilon \leq v_k \leq \bar{V}_k$$

$$F_{\min} \leq F_k \leq F_{\max}$$

$$P_{\min} \leq F_k v_k \leq P_{\max}$$

$$t_N - t_0 \leq T$$

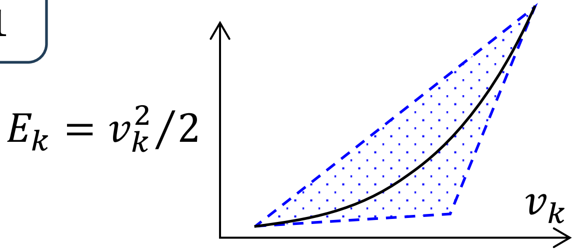
$$v_0 = V_0, v_N = V_f$$

$$z_k = 1/v_k \xrightarrow{\text{relaxed}} z_k \geq 1/v_k$$

$$E_k = v_k^2/2$$

Relaxed: $\mathrm{NLP}_{\mathrm{R2}}$

McCormick relaxation



$\mathrm{NLP}_{\mathrm{R2}}$ can be further sped up by adding two valid inequalities (VIs):

$$(+)\ P_{\min} z_k \leq F_k \leq P_{\max} z_k$$

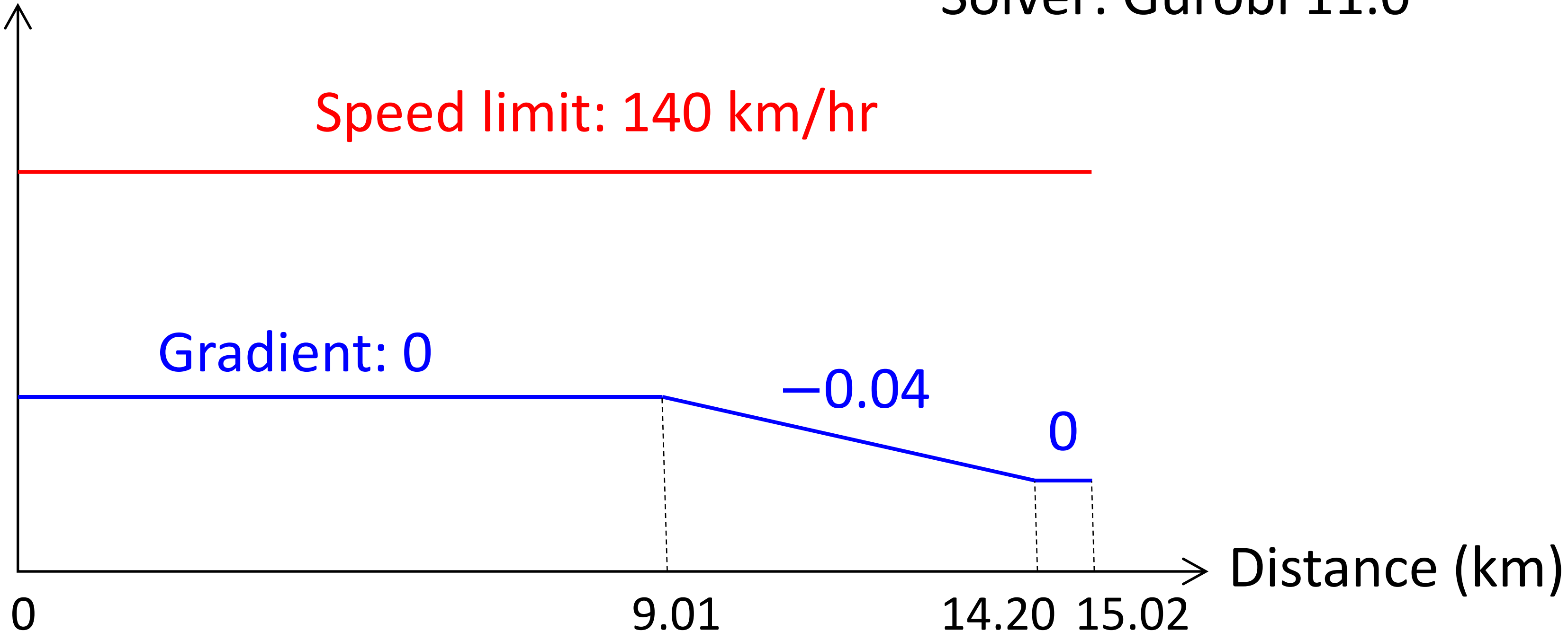
$$(+)\ E_k \geq v_k^2/2$$

$\mathrm{NLP}_{\mathrm{R2}}$ + VIs: $\mathrm{NLP}_{\mathrm{R3}}$

Case Study: Scenario

Scheduled trip time: 540 s

Solver: Gurobi 11.0



Case Study: Results

Instance	MILP with 32 pieces			NLP_{R1} (original)		NLP_{R2} (relaxed)		NLP_{R3} (R2+VIs)	
	Ctime (sec)	Gap (%)	Diff (%)	Ctime (sec)	Gap (%)	Ctime (sec)	Gap (%)	Ctime (sec)	Gap (%)
N = 170	146.43	0	1.21	3.01	0	2.71	0	0.89	0
N = 320	3600.69	1.7	-	8.54	0	5.65	0	2.68	0
N = 395	3600.57	-	-	135.25	0	10.53	0	3.32	0
N = 520				107.27	0	13.48	0	4.43	0
N = 770				3600.71	0.1	35.12	0	11.58	0
N = 1020				3600.55	0.1	80.69	0	19.29	0

Ctime: computing time

Gap: optimality gap when the solver terminated

Diff: difference of energy consumption from the NLP result

Conclusion, Extensions of Our Method, and Future Directions

- **Conclusion**: via reformulation, relaxation and valid inequalities, we developed a (more efficient) method of obtaining the exact solution to the nonlinear optimization problem for the classic single-train eco-driving problem
- We have **extended** our method to solve the eco-driving problems with:
  - Intermediate time-window constraints on a single train;
  - Joint eco-driving of a fleet of trains under green-wave policy for fixed-block signaling
- Possible **future directions**
  - More general signaling constraints: general fixed-block signaling with signal-dependent speed limit; moving-block; virtual coupling
  - Eco-driving under stochastic factors